22. A Note on the Large Sieve. III

By Yoichi MOTOHASHI
Department of Mathematics, College of Science and Technology, Nihon University

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1. The purpose of the present note is to prove a large sieve version of a recent sieve result of Selberg [4] by combining his argument with that of our preceding note [1] of this series.

Before stating our results we have to introduce some conventions: For a prime p let $\Omega(p^\alpha)$ be a set of residues (mod p^α), and let us assume that $\Omega(p^\alpha)$ and $\Omega(p^\beta)$ are disjoint (mod p^β) whenever $0<\beta<\alpha$. For a composite d $\Omega(d)$ denotes the set of residues (mod d) arising from those of $\Omega(p^\alpha)$ with $p^\alpha \| d$ (the maximum power of p dividing d), and we write $n \in \Omega(d)$ to indicate that n (mod p^α) $\in \Omega(p^\alpha)$ for each $p^\alpha \| d$; so $n \in \Omega(1)$ for any n.

Following Selberg we put

$$\theta(p^\alpha)=1-\sum_{j=1}^{\alpha}|\Omega(p^j)|p^{-j},$$
$$g(d)=d^{-1}\prod_{p^\alpha \| d}\{|\Omega(p^\alpha)|\theta(p^\alpha)/\theta(p^{\alpha-1})\},$$

$|\Omega(p^\alpha)|$ being the cardinality of the set; here and in what follows we may assume $\theta(p^\alpha)\neq 0$ always. Also, if $d|r$, we put

$$t(r,d)=\prod_{\substack{p^\alpha \| r\\ p^\beta \| d}} t(p^\alpha,p^\beta), \qquad t^*(r,d)=\prod_{\substack{p^\alpha \| r\\ p^\beta \| d}} t^*(p^\alpha,p^\beta),$$

where $t(p^\alpha,p^\beta)=1$ if $\alpha=\beta$, $=|\Omega(p^\alpha)|p^{-\alpha}$ if $\beta=0$, and $=-|\Omega(p^\alpha)|(\theta(p^\beta)p^\alpha)^{-1}$ if $0<\beta<\alpha$; $t^*(p^\alpha,p^\beta)=1$ if $\alpha=\beta$, $=-|\Omega(p^\alpha)|(\theta(p^{\alpha-1})p^\alpha)^{-1}$ if $\beta=0$, and $=|\Omega(p^\alpha)|(\theta(p^{\alpha-1})p^\alpha)^{-1}$ if $0<\beta<\alpha$. Further $\Gamma_r(n,\Omega)$ stands for the sum

$$\sum_{\substack{u|r\\ n\in\Omega(u)}} t^*(r,u)$$

which is equal to $t^*(r,1)$ if $n \notin \Omega(p^\beta)$ for each $p^\beta | r$, $(\beta>0)$.

Then our results are as follows:

Theorem. *Uniformly for any complex numbers a_n and for any $M,N,Q>0$, we have*

$$\sum_{\substack{qr\leqq Q\\ (q,r)=1}}{}' \sum_{\chi(\mathrm{mod}\, q)}{}^{*} \frac{q}{\varphi(q)g(r)}\left|\sum_{M<n\leqq M+N}\chi(n)\Gamma_r(n,\Omega)a_n\right|^2 \leqq (N+Q^2)\sum_{M<n\leqq M+N}|a_n|^2,$$

where φ is the Euler function, $\sum^*$ denotes a sum over primitive Dirichlet characters χ, and $\sum'$ indicates that r is restricted by $g(r)\neq 0$.

Corollary. *If $a_n=0$ whenever there exists a p^α such that $n\in\Omega(p^\alpha)$,*