21. On the Homogeneous Lüroth Theorem

By Shigeru IITAKA
Department of Mathematics, University of Tokyo

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§1. **Lüroth theorem.** *Let $f, g \in \boldsymbol{C}[X_1, \cdots, X_n]$ such that f is irreducible and suppose that polynomials g and f are algebraically dependent. Then g is a polynomial of f. In particular, if g is also irreducible, then $g=\alpha f+\beta$, α and $\beta \in \boldsymbol{C}$.*

The above statement is equivalent to the Lüroth theorem in the case of polynomials. For the sake of convenience, we begin by giving a proof to the above statement by using logarithmic genera [1].

Proof. Let $A^n = \operatorname{Spec} \boldsymbol{C}[X_1, \cdots, X_n]$, $\Gamma = \operatorname{Spec} \boldsymbol{C}[f, g]$, and $C = \operatorname{Spec} \boldsymbol{C}[f] \cong A^1$. Denoting by Γ' the normalization of Γ in A^n, we have the following diagram:

$$\begin{array}{ccc} A^n & \longrightarrow & C \\ \downarrow & \circlearrowleft & \uparrow \\ \Gamma' & \longrightarrow & \Gamma \end{array}$$

Diagram 1

Hence $\bar{g}(\Gamma') \leq \bar{q}(A^n)=0$. Since Γ' is normal, we have $\Gamma' \cong A^1$ by [3, Example 1]. This implies that $\Gamma' = \operatorname{Spec} \boldsymbol{C}[\theta]$, $\theta \in \boldsymbol{C}[X_1, \cdots, X_n]$. From the inclusions $\boldsymbol{C}[f] \subset \boldsymbol{C}[f, g] \subset \boldsymbol{C}[\theta]$, we infer readily that f is a poly nomial of θ. However, since f is irreducible, f is a linear form of 1 and θ, hence $\boldsymbol{C}[f, g]=\boldsymbol{C}[f]$. Q.E.D.

§2. **Quasi-Albanese maps of complements of $\boldsymbol{P}^n$.** Let $F_0, F_1, \cdots, F_r$ be mutually distinct (up to constant multiple) irreducible polynomials with $d_j = \deg F_j$. Consider a sublattice L of $\boldsymbol{Z}^{1+r}$ defined by

$$L=\{\boldsymbol{a} \in \boldsymbol{Z}^{1+r} ; \langle \boldsymbol{a}, \boldsymbol{d} \rangle = 0,\ \boldsymbol{d}=(d_0, \cdots, d_r)\}.$$

Let $(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_r)$ be a $\boldsymbol{Z}$-basis of L. Put

$$\Phi_j = \prod F_l^{m(l)}, \qquad \text{where } \boldsymbol{a}_j=(m(1), \cdots, m(r)).$$

Then we have a morphism

$$\alpha=(\Phi_1, \cdots, \Phi_r) : V=\boldsymbol{P}^n - \bigcup V_+(F_j) \longrightarrow \boldsymbol{C}^{*r}.$$

α coinsides with the quasi-Albanese map of V (cf. [2]). Denote by Δ the closed image of V by α. Δ is an affine variety whose coordinate ring $\Gamma(\Delta, \mathcal{O}_\Delta)$ is isomorphic to

$$\boldsymbol{C}[\Phi_1, \cdots, \Phi_r, \Phi_1^{-1}, \cdots, \Phi_r^{-1}].$$

Proposition 1. *Suppose that* $\dim \Delta=1$. *Then*

i) *Δ is non-singular,*

ii) *any general fiber of $\alpha: V \to \Delta$ is irreducible.*

Proof. This follows easily from the universality of quasi-Albanese