19. *Finitely Additive Measures on* N

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1. Introduction. In this paper, we improve the theorem of Jech and Prikry [2] on projections of finitely additive measures. Let N denote the set of all natural numbers. A (finitely additive) measure on N is a function $\mu: P(N)\to[0, 1]$ such that $\mu(\phi)=0$, $\mu(N)=1$ and if X and Y are disjoint subsets of N, then $\mu(X\cup Y)=\mu(X)+\mu(Y)$. μ is nonprincipal if $\mu(E)=0$ for every finite set $E\subset N$. Let $F: N\to N$ be a function. If μ is a measure on N, then $\nu=F^*(\mu)$ (the projection of μ by F) is the measure defined by $\nu(X)=\mu(F^{-1}(X))$.

Theorem (Jech and Prikry). *There exist a measure μ on N and a function $F: N\to N$ such that*

a) $F^*(\mu)=\mu$,

b) *if $X\subseteq N$ is such that F is one-to-one on X, then* $\mu(X)\leqq\dfrac{1}{2}$.

A measure is two-valued if the values is $\{0, 1\}$. The theorem of Jech and Prikry contrasts with the following theorem concerning two-valued measure (Frolik [1] and Rudin [3]):

If μ is a two-valued measure and $F: N\to N$ is such that $F^*(\mu)=\mu$, then $F(x)=x$ on a set of measure 1.

In this paper we prove the following

Theorem. *There exist a measure μ and a function $F: N\to N$ such that*

a) $F^*(\mu)=\mu$,

b) *if $X\subseteq N$ is such that F is one-to-one on X, then* $\mu(X)=0$.

2. Sketch of the proof. We shall now state two results, to be proved in the following sections. We shall indicate how Theorem follows from them.

Proposition 1. *For any prime p, there exist a function $F_p: N\to N$ and a finitely additive measure η_p such that*

1) $F_p^*(\eta_p)=\eta_p$,

2) *if $X\subseteqq N$ is such that F_p is one-to-one on X, then* $\eta_p(X)\leqq 1/(p-1)$.

Proposition 2. *There exists a function $f_p: N\xrightarrow[\text{onto}]{1;1}N$ such that $f_pF_3^{-1}=F_p^{-1}f_p$ where F_3 and F_p are the functions in Proposition* 1.

We let $F=F_3$ and $\lambda_p(X)=\eta_p(f_p(X))$ where $f_p(X)=\{f_p(x)|x\in X\}$.