11. A Remark on Convergence of Nonlinear Semigroups

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1. Introduction. Let X be a real Banach space. Let A_n, $n=1, 2, \cdots$, and A be *dissipative operators* in X which satisfy the conditions

$$R(I-\lambda A_n)\supset \overline{D(A_n)} \quad \text{and} \quad R(I-\lambda A)\supset \overline{D(A)} \qquad \text{for } \lambda>0.$$

Let $\{T_n(t)\,;\,t\geqq 0\}$ and $\{T(t)\,;\,t\geqq 0\}$ be the (*nonlinear*) *semigroups* generated by A_n and A in the sense of Crandall-Liggett [6]. It was shown by Brezis-Pazy [4] that *if* $\overline{D(A)}\subset\overline{D(A_n)}$, $n=1,2,\cdots$, *then the following property* (i) *implies the property* (ii).

(i) $$\lim_{n\to\infty}(I-\lambda A_n)^{-1}=(I-\lambda A)^{-1}$$

for each $\lambda>0$ *and* $x\in\overline{D(A)}$.

(ii) $$\lim_{n\to\infty} T_n(t)=T(t)x$$

for each $x\in\overline{D(A)}$ *and the limit is uniform on bounded t-intervals.*

Our aim in this note is to show that the property (ii) implies (i) under some additional conditions. Precisely, we shall show the following

Theorem. *Let* X^* *be uniformly convex. If* $\overline{D(A)}$ *is convex and* $\overline{D(A)}\subset\overline{D(A_n)}$, $n=1,2,\cdots$, *then the property* (ii) *implies the property* (i).

The above theorem is due to Bénilan [3] in the Hilbert space case. The idea of our proof of the theorem is essentially due to the recent work [1] of Baillon. As usual, we define the duality map F on X into X^* by $F(x)=\{x^*\in X^*\,;\,\langle x, x^*\rangle=\|x\|^2=\|x^*\|^2\}$. If X^* is uniformly convex, then F is single-valued and uniformly continuous on each bounded set of X. We refer to Barbu [2] for some properties of the duality map and nonlinear semigroups.

2. Proof of Theorem. Let $\overline{D(A)}$ be convex and $\overline{D(A)}\subset\overline{D(A_n)}$, $n=1,2,\cdots$, and assume the property (ii). Let $x\in\overline{D(A)}$ and $\lambda>0$ be fixed. We set $y_n=(I-\lambda A_n)^{-1}x$. We want to show that y_n converges to $(I-\lambda A_n)^{-1}x$ as $n\to\infty$. For the purpose, we prepare some lemmas.

Lemma 1. $\|y_n\|=O(1)$ *as* $n\to\infty$.

Proof. By Theorem 9 in [4], we have

$$\|y_n-x\|\leqq\frac{4}{\lambda}\int_0^\lambda\|T_n(\tau)x-x\|\,d\tau.$$

Since $T_n(\tau)x$ is bounded as $n\to\infty$ uniformly for $\tau\in[0,\lambda]$ by (ii), it follows that $\|y_n\|$ is bounded as $n\to\infty$. Q.E.D.

By the Hahn-Banach theorem, there exists a linear functional L