96. *On the Mordell-Weil Group of Certain Elliptic Curve*

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§1. **Introduction.** Let us consider the elliptic curve

(1.1) $E_j: y^2=x^3-(27j/4(j-1728))(x-1) \qquad (j\neq 0, 1728, \infty)$

which is a well known example of an elliptic curve defined over $\boldsymbol{Q}(j)$ with the absolute invariant j. For any value of j, the point $P_0: (x, y) = (1, 1)$ is a $\boldsymbol{Q}$-rational point of E_j. The purpose of this paper is to prove the following

Theorem 1.1. *For every $j \in \boldsymbol{Q}$ ($j\neq 0$, 1728), P_0 is a $\boldsymbol{Q}$-rational point of E_j of infinite order.*

Corollary 1.2. *For any $j \in \boldsymbol{Q}$, there exists an elliptic curve E defined over $\boldsymbol{Q}$ such that* 1) *the absolute invariant is j and* 2) *rank* $(E(\boldsymbol{Q}))\geq 1$.

The proof depends on the following remarkable theorem due to Barry Mazur:

Theorem 1.3 (Mazur [6]). *The order of a $\boldsymbol{Q}$-rational torsion point of an elliptic curve defined over $\boldsymbol{Q}$ is one of the following:*

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12\}.$$

For the proof of our theorem, we first show that, in case j is a variable over $\boldsymbol{Q}$, P_0 is a rational point of E_j of infinite order, by considering the associated elliptic surface over the j-line $\boldsymbol{P}^1$. Given a positive integer m, the set $A(m)$ of $j_0 \in \boldsymbol{Q}-\{0, 1728\}$ such that P_0 is a point of exact order m on E_{j_0} is obviously finite (cf. Proposition 3.2). Then by Mazur's theorem 1.3, $A(m)$ is empty if $m>12$ or $m=11$. Thus we have only to prove that $A(m)$ is also empty for $1\leq m\leq 10$ or $m=12$. This will be done case by case.

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§2. **Rational points on the generic fibre.** Put $t=27j/4(j-1728)$, then the equation of E_j becomes $y^2=x^3-tx+t$. From now on, we call this E_t. We note that if $j=0, 1728, \infty$, then $t=0, \infty, 27/4$, respectively. We note first the following

Proposition 2.1. rank $(E_t(\boldsymbol{Q}(t)))=1$, *where t denotes a variable over $\boldsymbol{Q}$.*

Proof. Let $B\xrightarrow{\Phi}\boldsymbol{P}^1$ be the elliptic surface associated to E_t, then we have (Shioda [9])