87. *Infinitely Divisible Distributions and Ordinary Differential Equations*

By Takesi WATANABE
Department of Mathematics, Osaka University

(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1979)

1. Introduction. In the study of the limit distributions of multi-type Galton-Watson processes, S. Sugitani [2] has discovered that if a nonnegative function $\psi(t, \lambda)$, defined for $t \geqq 0$, $\lambda \geqq 0$, satisfies the ordinary differential equation having λ as a parameter

$$(1)\qquad \psi' = -B\psi^2 + \lambda p(t), \qquad \psi(0, \lambda) = m\lambda,$$

where $B > 0$, $m \geqq 0$ and $p(t)$ is a polynomial with positive coefficients, then there exists for each $t > 0$ an infinitely divisible distribution ν_t on $[0, \infty)$ such that

$$(2)\qquad \exp\left\{-\int_0^t \psi(s, \lambda)ds\right\} = \int_0^\infty e^{-\lambda x}\nu_t(dx).$$

Further information on ν_t is given in [3].

In this note we will prove a stronger result that $\exp\{-\psi(t, \lambda)\}$ is the Laplace transform of some infinitely divisible distribution μ_t on $[0, \infty)$. Our proof is quite elementary and can be applied to more general equations.

2. A heuristic argument. Given $f(x)$, $g(t, \lambda)$ and $h(\lambda)$ defined for $x \in (-\infty, \infty)$, $t \in [0, T]$, $\lambda \in [0, \infty)$, consider the following ordinary differential equation having λ as a parameter;

$$(3)\qquad \psi' = f(\psi) + g(t, \lambda), \qquad \psi(0, \lambda) = h(\lambda).$$

For the moment, we assume that equation (3) has a unique solution $\psi(t, \lambda)$ in $[0, T] \times [0, \infty)$. Here and after we will write ψ' for $D_t\psi(t, \lambda)$, $f^{(n)}$ for $D_x^n f$, $g_{n\lambda}(t, \lambda)$ for $D_\lambda^n g(t, \lambda)$ and so on. We now seek a suitable condition in order that *$\psi_\lambda(t, \lambda)$ is completely monotonic in $\lambda \in (0, \infty)$ for each $t \geqq 0$.* The essential part of our condition is that *$-f^{(2)}(\cdot)$, $g_\lambda(t, \cdot)$ for each $t \in [0, T]$ and $h_\lambda(\cdot)$ are completely monotonic in $(0, \infty)$.* To show the above assertion, differentiating (3) with respect to λ, we have

$$\psi'_\lambda = f^{(1)}(\psi)\psi_\lambda + g_\lambda(t, \lambda), \qquad \psi_\lambda(0, \lambda) = h_\lambda(\lambda).$$

Since $g_\lambda(t, \lambda) \geqq 0$ and $h_\lambda(\lambda) \geqq 0$, it follows that $\psi_\lambda(t, \lambda) \geqq 0$. Similarly, n-times differentiation of (3) leads us to

$$\psi'_{n\lambda} = f^{(1)}(\psi)\psi_{n\lambda} + f^{(2)}(\psi) \sum_{\substack{k_1+k_2=n \\ 1 \leqq k_1 \leqq k_2}} c_{k_1,k_2}\, \psi_{k_1\lambda}\psi_{k_2\lambda}$$
$$+ \cdots + f^{(j)}(\psi) \sum_{\substack{k_1+\cdots+k_j=n \\ 1 \leqq k_1 \leqq \cdots \leqq k_j}} c_{k_1,\ldots,k_j}\, \psi_{k_1\lambda} \cdot \cdots \cdot \psi_{k_j\lambda}$$