71. J-Compatible Orthodox Semigroups

By Miyuki YAMADA
Department of Mathematics, Shimane University

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1979)

A semigroup S is said to be *J-compatible* if Green's J-relation is a congruence on S. In this paper, we shall study the structure of J-compatible orthodox semigroups.

1. Basic properties. Let S be a regular semigroup. Throughout this paper, the J-relation and the D-relation on S will be denoted by $\mathcal{J}_S$ and $\mathcal{D}_S$ respectively. Further, the congruence generated by $\mathcal{J}_S[\mathcal{D}_S]$ will be denoted by $\mathcal{J}_S^*[\mathcal{D}_S^*]$. Let η_S be the least semilattice congruence on S. Then, it has been shown by Hall [2] that $\mathcal{D}_S^*=\eta_S$. Further, it is easily seen that $\mathcal{D}_S\subset\mathcal{J}_S\subset\eta_S$. In fact, if $(a, b)\in\mathcal{D}_S$ then there exists $c\in S$ such that $a\mathcal{L}_S c\mathcal{R}_S b$, where $\mathcal{L}_S$ and $\mathcal{R}_S$ denote the L-relation and the R-relation on S respectively. Hence, $Sa=Sc$ and $cS=bS$. Accordingly, $SaS=ScS=SbS$. Therefore, $(a, b)\in\mathcal{J}_S$. Since η_S is the least semilattice congruence on S, S is a semilattice Γ of the η_S-classes $\{S_\gamma:\gamma\in\Gamma\}$ (in this case, $\Gamma\cong S/\eta_S$) and each S_γ is semilattice-indecomposable. If $a\mathcal{J}_S b$, then $SaS=SbS$. Hence, there exist $x, y\in S^1$ and $u, v\in S^1$ such that $uav=b$ and $xby=a$. Let $a\in S_\alpha$ and $b\in S_\beta$. Since $uav=b$, it follows that $\beta\leq\alpha$. On the other hand, $\alpha\leq\beta$ follows from $xby=a$. Hence, $\alpha=\beta$. Consequently, $a\eta_S b$. Thus, $\mathcal{J}_S\subset\eta_S$.

Since $\mathcal{D}_S^*=\eta_S$, we have $\mathcal{J}_S^*=\eta_S$. Hence, $\mathcal{D}_S^*=\mathcal{J}_S^*=\eta_S$. In particular, if S is J-compatible then $\mathcal{D}_S^*=\mathcal{J}_S=\eta_S$.

Theorem 1. *For a regular* [*orthodox*] *semigroup S, the following conditions are equivalent*:

(1) *S is J-compatible.*

(2) $\mathcal{J}_S=\eta_S$.

(3) *S is a semilattice of simple regular* [*orthodox*] *semigroups.*

(4) $J(a)\cap J(b)=J(ab)$ *for $a, b\in S$ (where $J(x)=SxS$); hence, the principal ideals of S form a semilattice under intersection.*

Proof. (1)$\Rightarrow$(2): This was already proved above. The part "(2)$\Rightarrow$(3)" follows from Petrich [6, p. 43]. Further, both "(3)$\Rightarrow$(4)" and "(4)$\Rightarrow$(1)" follow from Clifford and Preston [1, p. 123].

If $\mathcal{D}_S^*=S\times S$ for a semigroup S, then S is said to be *D^*-simple*.

Theorem 2. *If a regular semigroup S is simple, then S is D^*-simple.*

Proof. As was shown above, $\mathcal{D}_S^*=\eta_S$. Since S is simple, η_S is the universal relation. Hence, $\mathcal{D}_S^*$ is also the universal relation. That is,