37. A Version of the Central Limit Theorem for Martingales

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(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1979)

§ 1. **Introduction.** In the present note let $\{X_n, \mathcal{F}_n\}$ be a zero-mean square-integrable martingale on a probability space $(\Omega, \mathcal{F}, P)$ and let $Y_1 = X_1$, $Y_n = X_n - X_{n-1}$ $(n \geqq 2)$. Then our purpose is to prove the following

Theorem. *Suppose that there exist a sequence $\{A_n\}$ of positive numbers for which $\lim_{n\to+\infty} A_n = +\infty$ and a random variable $Z(\omega)$ such that*

(L-I) *for any given $\varepsilon > 0$, $\lim_{n\to+\infty} A_n^{-2} \sum_{k=1}^n E\{Y_k^2 I(|Y_k| \geqq \varepsilon A_n)\} = 0$,**)

(L-II) *$\lim_{n\to+\infty} A_n^{-2} \sum_{k=1}^n Y_k^2 = Z$, in probability.*

Then for any set $F \in \sigma(\bigcup_{n=1}^\infty \mathcal{F}_n)$ and any real number x $(x \neq 0)$

$$\lim_{n\to+\infty} P\{F, X_n(\omega)/A_n \leqq x\} = (2\pi)^{-1/2} \int_F \left\{\int_{-\infty}^{x/\sqrt{Z}} \exp(-u^2/2)du\right\} dP,$$

where $\sigma(\bigcup_{n=1}^\infty \mathcal{F}_n)$ denotes the σ-algebra generated by the algebra $\bigcup_{n=1}^\infty \mathcal{F}_n$ and $x/0$ is $+\infty$ (or $-\infty$) if x is positive (or negative).

In the important special case when Y_n's are independent and $\mathcal{F}_n$ is the σ-algebra generated by $\{X_k, k \leqq n\}$ the condition (L-I) for $A_n^2 = EX_n^2$ is called Lindeberg's condition for the central limit theorem and in this case (L-I) implies (L-II) with $Z(\omega) = 1$. But in general (L-I) does not imply (L-II) and even if the conditions (L-I) and (L-II) are satisfied the limit Z is not necessarily a constant. When $Z(\omega)$ is a constant, the central limit theorems are proved by many authors (cf. [1]).

As an application of Theorem we can prove the central limit theorem for $\{X_n\}$. In fact we prove the following

Corollary. *Under the conditions* (L-I) *and* (L-II) *if* $P\{Z(\omega) \neq 0\} > 0$, *then we have for any real number x*

$$\lim_{n\to+\infty} P\{X_n(\omega)/A_n \leqq x\sqrt{Z(\omega)} \mid Z(\omega) \neq 0\} = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-u^2/2)du.$$

In § 2 we prove Theorem. By Lévy's continuity theorem it is enough to show that, for any fixed real number λ,

$$\lim_{n\to+\infty} \int_F \exp(i\lambda X_n/A_n)dP = \int_F \exp(-\lambda^2 Z/2)dP. \tag{1.1}$$

The right hand side of the above formula is the Fourier-Stieltjes transform of the function $(2\pi)^{-1/2} \int_F \left\{\int_{-\infty}^{x/\sqrt{Z}} \exp(-u^2/2)du\right\} dP$, $-\infty < x$

*) $I(A)$ denotes the indicator of the set A.