75. On Vinogradov's Zero-Free Region for the Riemann Zeta-Function

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1. Let $\zeta(s)$ $(s=\sigma+it)$ be the Riemann zeta-function. And let $Y(t)=(\log(|t|+3))^{2/3}(\log\log(|t|+3))^{1/3}$. Also let c denote generally a positive absolute constant whose value may differ at each occurrence. Then, as is well-known, we have

Theorem 1. *$\zeta(s)$ does not vanish for $\sigma\geqq 1-cY(t)^{-1}$.*

Previous proofs of this fact are all dependent either on the theory of integral functions or on a function-theoretical lemma of Landau. The purpose of the present note is to show briefly that there exists still another proof which does not depend at all on the deep function-theoretical properties of $\zeta(s)$. Our main tools are the Vinogradov-Richert theorem (Lemma 1 below), the Selberg sieve and an argument closely related to that of [1].

As a by-product of our procedure we can prove also

Theorem 2. *Let U be sufficiently large, and let us assume*

$$\zeta(1+iU)^{-1}\ll D(U)(\log U)^{2/3}(\log\log U)^{1/4},$$

where $D(U)$ increases monotonically to infinity as $U\to\infty$. Then $\zeta(s)$ does not vanish for

$$\sigma\geqq 1-cY(U)^{-1}\log D(U),\quad |t|\leqq U/2,\quad t=\pm U.$$

The proof of Theorem 2 will not be given below; we mention only that it is derived from Lemmas 3 and 4. The detailed account will appear elsewhere.

2. Throughout in this and the next sections we assume that T is sufficiently large and that $1-\delta+iT$ is a zero of $\zeta(s)$ such that $0<\delta\leqq(\log T)^{-2/3}$. Because of the reason stated at the end of this section, we may presume also $\delta\geqq(\log T)^{-10}$.

Now let $\sigma(n;a)$ be the sum of the a-th powers of divisors of n, and let us put $f(n)=\sigma(n;-\delta-iT)$. We apply the Selberg sieve to the sequence $\{|f(n)^2|\}$. According to the general theory we should put

$$g(r)=\prod_{p|r}(F_p-1),\qquad G(R)=\sum_{r\leqq R}\mu^2(r)g(r),$$

where

$$F_p=\sum_{m=0}^{\infty}|f(p^m)|^2 p^{-m}.$$

Then the optimal weight is given by