69. *Some Properties of Non-Commutative Multiplication Rings*

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In this short note we shall discuss some properties of non-commutative multiplication rings, especially non-idempotent multiplication rings. Commutative multiplication rings were studied by S. Mori in [3], [4], and also in his earlier works. We denote $A \subseteq B$ if A is a subset of B, and by $A < B$ if A is a proper subset of B. We do not assume the existence of the identity, and "ideal" means a two-sided ideal.

1. Multiplication rings. Definition. A ring R is called a *multiplication ring* or briefly *M-ring*, if for any ideal $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{a} < \mathfrak{b}$, there exist ideals $\mathfrak{c}, \mathfrak{c}'$ such that $\mathfrak{a} = \mathfrak{b}\mathfrak{c} = \mathfrak{c}'\mathfrak{b}$.

Proposition 1. *Let R be an M-ring, let $\mathfrak{p}$ be a proper prime ideal, and let $\mathfrak{q}$ be any ideal properly containing $\mathfrak{p}$, then $\mathfrak{p}\mathfrak{q} = \mathfrak{q}\mathfrak{p} = \mathfrak{p}$.*

Proof. Since $\mathfrak{p} < \mathfrak{q}$, there exist ideals $\mathfrak{b}, \mathfrak{b}'$ such that $\mathfrak{p} = \mathfrak{q}\mathfrak{b} = \mathfrak{b}'\mathfrak{q}$, therefore $\mathfrak{p} \subseteq \mathfrak{b}$. On the other hand $\mathfrak{q}\mathfrak{b} \equiv 0 \pmod{\mathfrak{p}}$, $\mathfrak{q} \not\equiv 0 \pmod{\mathfrak{p}}$, implies $\mathfrak{b} \equiv 0 \pmod{\mathfrak{p}}$, hence $\mathfrak{p} = \mathfrak{b}$, and similarly $\mathfrak{p} = \mathfrak{b}'$.

Proposition 2. *Let R be an M-ring, and let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals such that $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$ and $\mathfrak{p}_2 \not\subseteq \mathfrak{p}_1$, then $\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_2\mathfrak{p}_1$.*

Proof. Since $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$, $\mathfrak{p}_2 < (\mathfrak{p}_1, \mathfrak{p}_2)$, therefore by Proposition 1 $\mathfrak{p}_2 = \mathfrak{p}_2(\mathfrak{p}_1, \mathfrak{p}_2) = (\mathfrak{p}_2\mathfrak{p}_1, \mathfrak{p}_2^2)$. If $\mathfrak{p}_2\mathfrak{p}_1 = \mathfrak{p}_1$, then we have $\mathfrak{p}_2 \supseteq \mathfrak{p}_1$, which contradicts our assumptions, therefore $\mathfrak{p}_2\mathfrak{p}_1 < \mathfrak{p}_1$, hence there exists an ideal $\mathfrak{c}$ such that $\mathfrak{p}_2 \supseteq \mathfrak{p}_2\mathfrak{p}_1 = \mathfrak{p}_1\mathfrak{c}$, and $\mathfrak{p}_1 \not\equiv 0 \pmod{\mathfrak{p}_2}$, therefore $\mathfrak{c} \equiv 0 \pmod{\mathfrak{p}_2}$. Thus we have $\mathfrak{p}_2\mathfrak{p}_1 \subseteq \mathfrak{p}_1\mathfrak{p}_2$. In a similar way we have $\mathfrak{p}_1\mathfrak{p}_2 \subseteq \mathfrak{p}_2\mathfrak{p}_1$, therefore $\mathfrak{p}_2\mathfrak{p}_1 = \mathfrak{p}_1\mathfrak{p}_2$.

Theorem 1. *Let R be an M-ring, then the multiplication of prime ideals is commutative.*

Proof. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals of R. If $\mathfrak{p}_1 < \mathfrak{p}_2$, then by Proposition 1 $\mathfrak{p}_1 = \mathfrak{p}_2\mathfrak{p}_1 = \mathfrak{p}_1\mathfrak{p}_2$. $\mathfrak{p}_2 < \mathfrak{p}_1$ implies the same results. If $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$ and $\mathfrak{p}_2 \not\subseteq \mathfrak{p}_1$, then by Proposition 2 $\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_2\mathfrak{p}_1$.

2. Non-idempotent M-ring. Definition. *An M-ring R such that $R > R^2$ is called a non-idempotent M-ring.*

Theorem 2. *Let R be non-idempotent M-ring, and let $\mathfrak{a}$ be an ideal of R, then $\mathfrak{a} = R^\rho$ for some positive integer ρ or $\mathfrak{a} \subseteq \bigcap_{n=1}^{\infty} R^n$.*

Proof. Let $\mathfrak{a}$ be an ideal such that $\mathfrak{a} \neq R^\rho$ for any positive integer ρ, then there exists n such that $\mathfrak{a} < R^n$, for example $n=1$, therefore $\mathfrak{a} = R^n\mathfrak{b}$ for some ideal $\mathfrak{b}$. Then $\mathfrak{a} = R^n\mathfrak{b} \subseteq R^nR = R^{n+1}$, and by our as-