55. *Asymptotic Behavior of Iterates of Nonexpansive Mappings in Banach Spaces*

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1. Introduction. Let X be a real Banach space and let X^* be the dual space of X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*). The *duality mapping* F (multi-valued) from X into X^* is defined by

$$F(x)=\{x^* \in X^* : (x, x^*)=\|x\|^2 \text{ and } \|x^*\|=\|x\|\} \qquad \text{for } x \in X.$$

We say that X is *smooth*, if $\lim_{t\to 0} t^{-1}(\|x+ty\|-\|x\|)$ exists for every x and y with $\|x\|=\|y\|=1$ (i.e., the norm of X is Gâteaux differentiable). It is shown that F is single-valued if and only if X is smooth. The duality mapping F of a smooth Banach space X is said to be *weakly continuous* at 0 if $w\text{-}\lim_{n\to\infty} x_n=0$ in X implies that $\{F(x_n)\}$ converges weakly* to 0 in X^*, where $w\text{-}\lim_{n\to\infty} x_n$ denotes the weak limit of $\{x_n\}$. It is easily seen that Hilbert space and (l^p), $1<p<\infty$, have this property.

Throughout the rest of this paper we assume that X is a smooth and uniformly convex real Banach space having the duality mapping F which is weakly continuous at 0, and C is a nonempty closed convex subset of X. A mapping $T: C \to C$ is said to be *nonexpansive* on C, or $T \in \mathrm{Cont}(C)$ if $\|Tx-Ty\| \leqq \|x-y\|$ for all $x, y \in C$. The set of fixed-points of T will be denoted by $\mathcal{F}(T)$, i.e., $\mathcal{F}(T)=\{x \in C: Tx=x\}$.

The purpose of this note is to prove the following

Theorem. *Let $T \in \mathrm{Cont}(C)$ and $x \in C$. The following three conditions are mutually equivalent:*

(i) $w\text{-}\lim_{n\to\infty} T^n x$ *exists*;

(ii) $\mathcal{F}(T) \neq \phi$ *and* $\omega_w(x) \subset \mathcal{F}(T)$;

(iii) $E(x) \neq \phi$ *and* $\omega_w(x) \subset E(x)$;

where $\omega_w(x)$ denotes the set of weak subsequential limits of $\{T^n x\}$, and $E(x)=\{u \in C: \|T^n x-u\|$ converges as $n\to\infty\}$. Moreover, if $w\text{-}\lim_{n\to\infty} T^n x$ exists, then it is the asymptotic center of $\{T^n x\}$ with respect to C.

In Hilbert space, the equivalence of (i) and (ii) in Theorem has been established by A. Pazy [5]. As corollaries of Theorem, we have the following:

Corollary 1 (Z. Opial [4]). *Let $T \in \mathrm{Cont}(C)$ and $x \in C$. If $\mathcal{F}(T) \neq \phi$ and $\|T^{n+1}x-T^n x\| \to 0$ as $n\to\infty$, then the sequence $\{T^n x\}$ is weakly convergent to an element of $\mathcal{F}(T)$.*