61. Energy Decay of Solutions of Dissipative Wave Equations

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(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1977)

1. Introduction. We shall investigate the energy decay of the solutions to the following Cauchy problem;

$$(1)\qquad \begin{cases} L(u)=u_{tt}-\Delta u+a(x,t)u_t=0, & x\in R^n,\ t\geq 0,\\ u(x,0)=\phi(x)\in C_0^\infty, & u_t(x,0)=\psi(x)\in C_0^\infty, \end{cases}$$

where $a(x,t)\in \mathscr{B}^1$ [*)], $a(x,t)\geq 0$ and Δ=Laplacian in R^n. Rauch and Taylor [3] showed that, if $a(x,t)\equiv a(x)$ and $a(x)$ has compact support, the energy $E(t)$ defined by

$$E(t)=\int_{R^n}|u_t(t)|^2+|\nabla u(t)|^2\,dx \qquad (\nabla\text{; gradient in } R^n)$$

for the solutions of (1) does not decay as t goes to infinity. More generally, Mochizuki [2] showed that, if $0\leq a(x,t)\leq c(1+|x|)^{-1-\delta}$ for some positive constants c and δ $(n\neq 2)$, $E(t)\not\to 0$ as $t\to+\infty$. On the other hand, we have from the usual energy estimates that if $a(x,t)\geq$ Const. >0 and $a_t(x,t)\leq 0$, $E(t)$ decays like $0(t^{-1})$. In this paper we give more general conditions which guarantee the decay of $E(t)$ and an application to the nonlinear wave equations. Now, letting m be a positive constant, we list up the assumptions:

(A–1) There exist some positive constants r, K and ε such that

$$\operatorname{supp}\phi(x)\cup\operatorname{supp}\psi(x)\subset\{x\in R^n\,|\,|x|\leq r\},$$

$$\min_{|x|\leq mt+r} a(x,t)\geq (K+\varepsilon t)^{-1} \qquad \text{for all } t\geq 0,$$

$$\max_{|x|\leq mt+r} a_t(x,t)\leq \varepsilon^2(2\gamma^2+6\gamma+3)(2+\gamma)^{-1}(K+\varepsilon t)^{-2} \qquad \text{for all } t\geq 0$$

where $\gamma=(3\varepsilon-2+\sqrt{9\varepsilon^2-4\varepsilon+4})/2$.

(A–2) $a(x,t)$ belongs to $\mathscr{B}^{k+1}$ $(k=1,2,\cdots)$ and satisfies

$$\max_{|x|\leq mt+r}\sum_{i=1}^{k}\left|\left(\frac{\partial}{\partial t}\right)^i a(x,t)\right|\leq \text{Const.}(1+t)^{-1} \qquad \text{for all } t\geq 0.$$

(A–3) $a(x,t)\equiv(K+\varepsilon t)^{-1}$ for some positive constants K and ε.

Then we have the following

Theorem 1. *Suppose* (A–1) *with* $m=1$. *Then the energy* $E(t)$ *for the solutions of* (1) *decays like* $0(t^{-2/(2+\gamma)})$. *Furthermore suppose* (A–2) (*resp.* (A–3)) *with* $m=1$. *Then the solutions of* (1) *satisfy*

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*) $\mathscr{B}^k$ is the set of all functions defined on $R^n\times[0,+\infty)$ such that all their partial derivatives of order $\leq k$ exist and are continuous and bounded.