60. *On the Least Positive Eigenvalue of the Laplacian for Riemannian Manifolds*

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§1. **Preliminaries.** Let M be an n-dimensional compact connected manifold. For a Riemannian metric g, let $-\Delta_g$ be the Laplacian associated to g acting on smooth functions on M. We may use the convention that the set of non-zero eigenvalues of Δ_g consists of the eigenvalues repeated a number of time equal to their multiplicities. For a fixed positive integer k, let $\lambda_1(g), \cdots, \lambda_k(g)$ be k eigenvalues chosen as small as possible. We consider the function Ξ_k on the space of smooth Riemannian metrics on M (cf. [1] p. 143):

$$\Xi_k(g)=V_g^{-2/n}\sum_{i=1}^{k}\lambda_i(g)^{-1},$$

where V_g is the volume of (M, g). For a fixed Riemannian metric g_0, let $m(g_0)$ be the multiplicity of the least positive eigenvalue $\lambda_1(g_0)$ of Δ_{g_0}. The function $\Xi=\Xi_{m(g_0)}$ is called (cf. [1]) to be critical at g_0 if

$$\left[\frac{d}{dt}\Xi(g(t))\right]_{t=0}=0,$$

for every one-parameter family of Riemannian metrics $g(t)$, $g(0)=g_0$, $|t|<\varepsilon$, depending real analytically on t.

§2. **Statements of Results.** Let K be a compact connected Lie group, K_0 a closed subgroup of K and $M=K/K_0$ the quotient manifold. Let g_0 be a K-invariant Riemannian metric on M. Then we have the following results:

Theorem 1. *Let $M=K/K_0$ be as above. Suppose that the linear isotropy representation of K_0 is irreducible over $\boldsymbol{R}$. Then the function Ξ is critical at the K-invariant metric g_0.*

Theorem 2. *Let $M=K/K_0$ be the compact homogeneous space of* $\dim. M\geq 2$. *In case of* $\dim. M>2$, *we assume the linear isotropy representation of K_0 is irreducible over $\boldsymbol{R}$. Let g_0 be a K-invariant metric on M. Then*

$$\Xi(\varphi g_0)\geq m(g_0)^{2/n-1}\Xi(g_0),$$

for every positive valued smooth function φ on M such that $\langle\varphi^{n/2},\eta\rangle_{g_0}=0$ for every $\eta\in\mathfrak{F}$. Here $\langle\cdot,\cdot\rangle_{g_0}$ is the L_2-inner product on the space of smooth functions on M and $\mathfrak{F}$ is the $\lambda_1(g_0)$ eigenspace of Δ_{g_0}.

Remark 1. The function $\varphi^{n/2}$ in Theorem 2 is given as follows, for example: Let ψ be a smooth function orthogonal to $\mathfrak{F}$ with respect