34. *A Note on the Large Sieve. II*

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1. Let $\mathcal{N}$ be a set of integers in an interval of length N. Also let $\mathcal{P}$ be a set of prime numbers p to each of which Ω_p a set of residues (mod p) is associated. It is assumed that $|\Omega_p|$ the number of elements of Ω_p satisfies $0<|\Omega_p|<p$. Then the large sieve under the present consideration is the problem of estimating

$$S=|\{n\in\mathcal{N}\,;\, n(\mathrm{mod}\,p)\notin\Omega_p \quad \text{for all } p\in\mathcal{P}\}|.$$

According to the famous theorem of Montgomery [2] (with the latter refinement [3]) we have

$$S\leqq(N+Q^2)\left\{\sum_{q\in\mathcal{Q}}\prod_{p|q}\frac{|\Omega_p|}{p-|\Omega_p|}\right\}^{-1} \tag{1}$$

where

$$\mathcal{Q}=\left\{q\leqq Q\,;\, q\,\Big|\prod_{p\in\mathcal{P}}p\right\}.$$

Kobayashi [1] made an important observation that the optimal value of the Selberg λ_d (see (2) below) can be put into an expression which combines well with the dual form of the (additive) large sieve inequality, and thus he got a proof of (1) via Selberg's procedure.

The purpose of the present note is to show that there is a simpler modification of Selberg's argument than Kobayashi's which leads us to (1) quite straightforwardly. In particular we do not need the explicit value of λ_d. But as [1] we have to appeal to the following result due to Montgomery and Vaughan [3; formula (2.3)]:

Lemma. *Let $\{x_j\}$ be a set of real numbers which are δ well-spaced* (mod 1). *Then, for any complex numbers b_j and real M and $N(>0)$, we have*

$$\sum_{M<n\leqq M+N}\left|\sum_j b_j e^{2\pi i x_j n}\right|^2\leqq(N+\delta^{-1})\sum_j|b_j|^2.$$

2. In order to simplify the notations we introduce the following conventions that $\Omega_d=\Omega_{p_1}\times\Omega_{p_2}\times\cdots\times\Omega_{p_t}$ if $\mathcal{Q}\ni d=p_1p_2\cdots p_t$ and that $n\in\Omega_d$ means n (mod d) $\in\Omega_d$, so $n\in\Omega_1$ for any n.

Then by the fundamental idea of Selberg we have

$$S\leqq\sum_{M<n\leqq M+N}\left|\sum_{n\in\Omega_d}\lambda_d\right|^2, \tag{2}$$

where $\mathcal{N}\subseteq(M,M+N]$ and λ_d are complex numbers defined on $\mathcal{Q}$ whose values are arbitrary, except for