33. A Counterexample to a Conjecture By P. Erdös

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1. Ch. Pommerenke [4] proved the following theorem. *Let* $f(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_{n-1}z+a_n$ *be a polynomial of degree* n *with some* $a_j\neq 0$. *Assume that the region* $E_f=\{z\in \boldsymbol{C}:|f(z)|\leqq 1\}$ *is connected, where* $\boldsymbol{C}$ *stands for the field of complex numbers. Then*

$$\max_{z\in E_f}|f'(z)|<\frac{en^2}{2}.$$

P. Erdös [5] reviewing Pommerenke's paper conjectured that

$$\max_{z\in E_f}|f'(z)|<\frac{n^2}{2}$$

is also true and it is best possible. Erdös reposed his conjecture as a problem in [2]. As it appears in [3] Erdös' conjecture was unsolved until the year 1972 and to the best of our knowledge it is open until now. The purpose of this paper is to give a counterexample to Erdös' conjecture. It seems to us that this gives some information concerning the famous coefficient conjecture of L. Bieberbach [1], [6], [7].

2. Counterexample to Erdös' conjecture. Let $T_n(z)$ be the Chebyshev polynomial of degree n, defined by $T_n(z)=2\cos n\theta$, where $z=2\cos\theta$, and $n=0,1,2,3,\cdots$. This is a complex polynomial of a real variable and has n real zeros in the line segment $[-2,2]$ and $-2\leq T_n(z)\leq 2$ for $-2\leq z\leq 2$. The recursion formula, $T_{n+1}(z)=zT_n(z)-T_{n-1}(z)$, which is valid since $\cos(n+1)\theta+\cos(n-1)\theta=2\cos n\theta\cos\theta$, allows us to write the following sequence of polynomials: $T_0(z)=2$, $T_1(z)=z$, $T_2(z)=z^2-2$, $T_3(z)=z^3-3z$, $T_4(z)=z^4-4z^2+2$ and in general

$$T_n(z)=z^n+\sum_{m=1}^{[n/2]}(-1)^m\frac{n}{m}\binom{n-m-1}{m-1}z^{n-2m}$$

is a complex inhomogeneous polynomial in a real variable and of degree n. Consider now $f(z)=\lambda^nT_n(z/\lambda)$. This is a monic inhomogeneous polynomial of degree n and in fact $-2\lambda^n\leq f(z)\leq 2\lambda^n$ for $-2\lambda\leq z\leq 2\lambda$. Take $\lambda=1/2^{1/n}$. Then $-1\leq f(z)\leq 1$ for $-2/2^{1/n}\leq z\leq 2/2^{1/n}$. Because of the fact that $T_n(z)=T_n(2\cos\theta)=2\cos n\theta$, it implies that $T'_n(2\cos\theta)=n(\sin n\theta/\sin\theta)$. Thus, $\max\{|T'_n(z)|: -2/2^{1/n}\leq z\leq 2/2^{1/n}\}=n^2$ because $\max\{(\sin n\theta/\sin\theta): -2/2^{1/2}\leq z\leq 2/2^{1/n}\}=n$. However, $f(z)=\lambda^nT_n(z/\lambda)$. Therefore $f'(z)=\lambda^{n-1}T'_n(z/\lambda)$ and so $\max\{|f'(z)|: -2\lambda\leq z\leq 2\lambda\}=\lambda^{n-1}n^2$. If we set $\lambda=1/2^{1/n}$, then $\max\{|f'(z)|: -2/2^{1/n}\leq z\leq 2/2^{1/n}\}$