31. A Note on Steenrod Operations in the Eilenberg-Moore Spectral Sequence

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1. Introduction and results. Let X be an associative H-space and BX the classifying space of X. The purpose of this note is to describe two kinds of Steenrod operations in the Eilenberg-Moore spectral sequence $\{E_r\}$ such that

$$E_2 \cong \mathrm{Cotor}_{H^*(X;\,Z_p)}(Z_p, Z_p) \Rightarrow H^*(BX; Z_p),$$

where p is a prime.

Our results are stated as follows.

Theorem 1. *In the Eilenberg-Moore spectral sequence $\{E_r\}$ there are Steenrod operations*

$$\beta^\varepsilon \mathcal{P}^i : E_r^{s,t} \to E_r^{s,t+2i(p-1)+\varepsilon}, \qquad 2i \le t,$$

and

$$\beta^\varepsilon \mathcal{P}^i : E_r^{s,t} \to E_r^{s+(2i-t)(p-t)+\varepsilon,pt}, \qquad 2i \ge t,$$

where $r \ge 2$ and $\varepsilon = 0$ or 1.

Remark. If $p=2$, we understand $\mathcal{P}^i = Sq^{2i}$ and $\beta\mathcal{P}^i = Sq^{2i+1}$.

Theorem 2. *Let $u \in E_r^{s,t}$.*

(i) *If $2i \le t-r+1$, then $d_r\beta^\varepsilon\mathcal{P}^i u = (-1)^\varepsilon \beta^\varepsilon \mathcal{P}^i d_r u$.*

(ii) *If $t-r+1 \le 2i \le t$, then $\beta^\varepsilon\mathcal{P}^i u$ survives to $E_q^{s,t+2i(p-1)+\varepsilon}$, where $q = r+(2i-t+r-1)(p-1)+\varepsilon$, $\beta^\varepsilon\mathcal{P}^i d_r u$ survives to $E_q^{s+q,t+2i(p-1)+\varepsilon+q-1}$, and*

$$d_q\beta^\varepsilon\mathcal{P}^i u = (-1)^\varepsilon\beta^\varepsilon\mathcal{P}^i d_r u.$$

(iii) *If $2i \ge t$, then $\beta^\varepsilon\mathcal{P}^i u$ survives to $E_q^{s+(2i-t)(p-1)+\varepsilon,pt}$, where $q = rp - p + 1$, $\beta^\varepsilon\mathcal{P}^i d_r u$ survives to $E_q^{s+(2i-t)(p-1)+\varepsilon+q,pt+q-1}$, and*

$$d_q\beta^\varepsilon\mathcal{P}^i u = (-1)^\varepsilon\beta^\varepsilon\mathcal{P}^i d_r u.$$

Theorem 3. *Let $p : F^{s,t} = F^{s,t}H^{s+t}(BX; Z_p) \to E_\infty^{s,t}$ be the natural projection and $u \in F^{s,t}$.*

(i) *If $2i \le t$, then $\beta^\varepsilon\mathcal{P}^i u \in F^{s,t}$ and $p\beta^\varepsilon\mathcal{P}^i u = \beta^\varepsilon\mathcal{P}^i pu$.*

(ii) *If $2i \ge t$, then $\beta^\varepsilon\mathcal{P}^i u \in F^{s+(2i-t)(p-1)+\varepsilon,t-(2i-t)(p-1)-\varepsilon}$ and $p\beta^\varepsilon\mathcal{P}^i u = \beta^\varepsilon\mathcal{P}^i pu$.*

Let $A = H^*(X; Z_p)$. It is well known that two kinds of Steenrod operations are defined on $\mathrm{Cotor}_A(Z_p, Z_p)$, that is, the vertical Steenrod operations

$$\beta^\varepsilon\mathcal{P}_V^i : \mathrm{Cotor}_A^{s,t} \to \mathrm{Cotor}_A^{s,t+2i(p-1)+\varepsilon}, \qquad 2i \le t,$$

and the diagonal Steenrod operations

$$\beta^\varepsilon\mathcal{P}_D^i : \mathrm{Cotor}_A^{s,t} \to \mathrm{Cotor}_A^{s+(2i-t)(p-1)+\varepsilon,pt}, \qquad 2i \ge t,$$

which satisfy the usual properties such as Cartan formula and Adem