18. On a Theorem Concerning the Homological Structure and the Holonomy Groups of Closed Orientable Symmetric Spaces.

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1. In his interesting manuscript "On the relation between homological structure of Riemannian spaces and exact differential forms which are invariant under holonomy groups" [6][1] written in Japanese, the late Mr. Iwamoto proved the following theorem: "Let B_p be the pth Betti number of a closed orientable Riemannian manifold M_n and B'_p the maximum number of linearly independent (in the sense of algebra) differential forms of rank p which are invariant under the holonomy group h of M_n, then $B_p \geqq B'_p$". As the skew symmetric tensors which are coefficients of differential forms Π invariant under the holonomy group h are covariant constant, Π's are harmonic differential forms. The above theorem is an immediate consequence of Hodge's theorem [5], for two distinct harmonic differential forms of rank p cannot be homologus.

In connexion with the above theorem, he stated without any indication of the proof the following:

Theorem: *If the Riemannian manifold in consideration is symmetric in the sense of Cartan, then,* $B_p = B'_p$.

The purpose of this paper is to give the proof of this theorem.

2. We shall start with the group theoretical definition of symmetric Riemannian spaces.

Let M_n be an n-dimensional homogeneous space with the Lie group of structure G and O be a point of M_n. Then all transformations of G which leave O unaltered constitute the group of isotropy g of M_n. Now, a one to one mapping π of G (as a topological space) onto itself which satisfies the properties (i) $\pi^2 = 1$ (involutive property), (ii) conservation of the law of composition, is called an involutive automorphism of G. It is evident that all elements of G which are invariant under π constitute a group, we shall call it the characteristic subgroup of G with respect to π. If the characteristic subgroup of G with respect to π coincides with the group of isotropy g, then we call M_n a symmetric space.

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1) The brackets [ ] denote the order of papers arranged in the bibliography at the end of this paper.