29. On a Topological Method in Semi-Ordered Linear Spaces.

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In Banach spaces, we always obtain a continuous linear functional as the limit of a weakly converging sequence of continuous linear functionals. And this property is based on a fact that a complete metric space is of second category. In continuous semi-ordered linear spaces, bounded (continuous or universally continuous) linear functionals have the same property[1]. To investigate the relation of these two cases, first in § 1 we will define a kind of topology in abstract spaces by which we obtain a topological space having the property akin to that of second category one under some condition. In § 2 applying it to semi-ordered linear spaces we will show that we can discuss the problem mentioned above by the topological method.

We shall make use of notations in the books of H. Nakano[2].

§ 1. Cell-topology.

Let R be an abstract space. For a family $\mathfrak{L}$ of subsets of R we denote by $\bar{\mathfrak{L}}$ the least totally aditive family including $\mathfrak{L}$ and the null set 0, and by $\mathfrak{T}$ the family of all the set X such that $C \in \bar{\mathfrak{L}}$ implies $XC \in \bar{\mathfrak{L}}$. Then we can see easily that $\mathfrak{T}$ satisfies the topological conditions[3] and hence we obtain a topology in R by which the family of all the open sets coincides with $\mathfrak{T}$. For brevity we will call it the topology by a *cell-system* $\mathfrak{L}$ and a set belonging in $\mathfrak{L}$ a *cell*. If $\mathfrak{L}$ satisfies the following condition:

$$(1) \quad \mathfrak{L} \ni C_\nu \ (\nu = 1, 2. \dots) \ C_1 \supset C_2 \supset \cdots, \text{ implies } \prod_{\nu=1}^{\infty} C_\nu \neq 0,$$

then a cell system $\mathfrak{L}$ is said to be *complete*.

Let R be a topological space by a complete cell-system $\mathfrak{L}$ in the sequel. Then R has the following important property:

Theorem 1.I. *For the sequence of closed sets* $B_\nu \ (\nu = 1, 2. \dots)$, *if every* B_ν *includes no cells, then the union* $\sum_{\nu=1}^{\infty} B_\nu$ *also includes no cells.*

Proof. If $\sum_{\nu=1}^{\infty} B_\nu \supset C \in \mathfrak{L}$ then there exist $C_\nu \in \mathfrak{L} \ (\nu = 1, 2. \dots)$ such that $B_1' C \supset C$, $B_2' C_1 \supset C_2 \dots B_\nu' C_{\nu-1} \supset C_\nu \dots$ because B_ν' is open and $\bar{\mathfrak{L}} \ni B_\nu' C_{\nu-1} \neq 0$. Therefore by (1) we obtain that $0 \neq C \prod_{\nu=1}^{\infty} C_\nu \subset C \prod_{\nu=1}^{\infty} B_\nu' = C(\sum_{\nu=1}^{\infty} B_\nu)'$ and come to the contradiction.