48. On the Notion of Measurability.

By Shizu ENOMOTO.

(Comm. by K. KUNUGI, M.J.A., May 16, 1951.)

Our main purpose of this paper is to give another definition of "Measurability" with respect to Carathéodory's outer measure than that which is given by Carathéodory himself.

Let X be a metric space. In this paper we shall consider Carathéodory's outer measure[1] μ defined for all sub-sets of X which satisfies the following condition:

(1) $\mu(A) < +\infty$ for every bounded set A,

and call it *conditionally finite outer measure.* The class of sets that are measurable (with respect to μ) in the sense of Carathéodory is denoted by $\mathfrak{C}(\mu)$.

Consider a class $\mathfrak{K}$ of sets which consists of the elements of $\mathfrak{C}(\mu)$ and in which there exists a sequence $\{K_n\}$ of bounded subsets such that $\bigcup_{n=1}^{\infty} K_n = X$ and $K_n \subseteq K_{n+1}$ for every n. Let $\mathfrak{R}(\mathfrak{K}, \{K_n\}, \mu)$ be the class of sets A satisfying $\mu(A_n) = \mu(K_n) - \mu(K_n \frown CA_n)$[2] for every n, where A_n is the common part A and K_n. For instance, for $\mathfrak{K}$ and $\{K_n\}$ we can take the class of circles $S_n(p)$ of n-radius with a fixed point p. We can see at once that the class $\mathfrak{R}(\mathfrak{K}, \{K_n\}, \mu)$ is independent on $\mathfrak{K}$ and $\{K_n\}$ and therefore we denote it by $\mathfrak{R}(\mu)$ and call *μ-modular set* the element of $\mathfrak{R}(\mu)$. When $\mathfrak{M}$ is a completely additive class and μ is completely additive on $\mathfrak{M}$, we say that $\mathfrak{M}$ is *μ-completely additive class.* Given classes of sets $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $[\mathfrak{M}, \mathfrak{N}]$ the smallest completely additive class containing $\mathfrak{M}$ and $\mathfrak{N}$.

Definition. Let μ be conditionally finite outer measure given in a metric space X. Let m be an arbitrary regular[3] outer measure satisfying the property such that $m(K_n) = \mu(K_n)$ for every K_n and that $m(A) \geq \mu(A)$ for the other sets A. Then, m is termed *dominant measure of μ* and any set A of $\mathfrak{R}(m)$ is said to be *m-dominant measurable set of μ.* Particularly, consider the set function $\mu_0(A) = \inf m_\alpha(A)$, where the infimum is taken over all dominant measure m_α of μ. In this case, if μ_0 is a regular outer measure and therefore a dominant measure of μ, the outer measure μ is called *relatively regular* and any μ_0-dominant measurable set of μ is called *μ-measurable set.*

1) C. Carathéodory: Vorlesungen über reelle Funktion (1927), § 235.
2) CA is complement of A.
3) C. Carathéodory: Loc. cit. § 253.