59. On a Theorem of Minkowski and Its Proof of Perron.

By Zuiman YÛJÔBÔ.

Department of Mathematics, St. Paul University.

(Comm. by Z. SUETUNA, M.J.A., June 12, 1951.)

Concerning the Diophantine approximation, there is a following theorem of Minkowski:

Theorem. For arbitrary two linear forms

$$L_1(x, y) = \alpha x + \beta y - \sigma, \quad L_2(x, y) = \gamma x + \delta y - \tau \qquad \left(\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \Delta \neq 0 \right)$$

there exists at least a lattice point (x, y) *which satisfies*

$$|L_1(x, y) L_2(x, y)| \leqq \frac{|\Delta|}{4}.$$

I will show in this paper that this can be improved as follows from its simple proof due to Perron.[1)]

Theorem. Under the same condition as above, there exist infinitely many lattice points (x_n, y_n) $(n = 1, 2, \dots)$ which satisfy $|x_n| \to \infty$, $|y_n| \to \infty$ and $|L_1(x_n, y_n) L_2(x_n, y_n)| \leqq \frac{|\Delta|}{4}$ with the inequalities $|L_1(x_n, y_n)| > K|x_n|$ and $> K|y_n|$, where K is a positive constant depending only on L_1 and L_2, if $\Delta \neq 0$, $\gamma, \delta \neq 0$ hold, γ/δ is not a rational number and $L_2(x, y) = 0$ has no lattice solution.

The particular case of this theorem, in which $L_1(x, y) = x$ and $L_2(x, y) = \theta x - y - \vartheta$ is already found by Minkowski too, and proved also by Koksma[2)] by using Perron's method.

Now let us explain our proof of the above theorem which is deduced from that proof of Perron and furthermore a proof of Korkine-Zortaroff-Markoff's theorem also due to Perron.[3)]

Without loss of generality we may consider the case, in which

$$L_1(x, y) = \alpha(x-\mu) + \beta(y-\nu), \quad L_2(x, y) = \gamma(x-\mu) + \delta(y-\nu). \qquad \left(\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \pm 1 \right)$$

---

1) O. Perron: Neuer Beweis eines Satzes von Minkowski. Math. Ann. 115 (1938).

2) J. F. Koksma: Anwendung des Perronschen Beweis eines Satzes von Minkowski. Math. Ann. 116, (1939).

3) O. Perron: Eine Abschätzung für die untere Grenze der absoluten Beträge der durch eine reelle oder imaginäre binäre quadratische Form darstellbaren Zahlen. Math. Zeits. 35 (1932).