76. On Selberg's Elementary Proof of the Prime-Number Theorem.

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Small Latin characters except x denote natural numbers; p represents a prime, and x denotes a real number $\geqq 1$.

A. Selberg obtained recently an elementary proof of the prime-number theorem using the following asymptotic formula:

$$\vartheta(x)\log x+\sum_{p\leq x}\vartheta\left(\frac{x}{p}\right)\log p = 2x\log x+O(x), \tag{1}$$

where

$$\vartheta(x)=\sum_{p\leq x}\log p.$$

We shall give in this note a simple proof above for

$$\psi(x)\log x+\sum_{n\leq x}\psi\left(\frac{x}{n}\right)\Lambda(n)=2x\log x+O(x), \tag{2}$$

where we define

$$\psi(x)=\sum_{n\leq x}\Lambda(n), \quad \Lambda(n)=\begin{cases}\log p & \text{for } n=p^l, \\ O & \text{otherwise.}\end{cases}$$

The formula (2) is as effective as (1) and may be used as a substitute for (1) in the proof of the prime-number theorem. (Of course we could prove directly, if we wished, the equivalence of the two formulae.)

We have clearly

$$\log n=\sum_{d\mid n}\Lambda(d), \tag{3}$$

and hence, by Möbius' inversion-formula,

$$\Lambda(n)=\sum_{d\mid n}\mu(d)\log\frac{n}{d}. \tag{4}$$

We find further, using (3),

$$\begin{aligned}\sum_{n\leq x}\psi\left(\frac{x}{n}\right)&=\sum_{mn\leq x}\Lambda(m)=\sum_{n\leq x}\sum_{d\mid n}\Lambda(d)\\ &=\sum_{n\leq x}\log n=\int_1^x\log\xi\,d\xi+O(\log x)\\ &=x\log x-x+O(\log x),\end{aligned} \tag{5}$$