137. On Completely Additive Classes of Sets with Respect to Carathéodory's Outer Measure.

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The purpose of this paper is to investigate the relations between completely additive classes of sets with respect to Carathéodory's outer measure. This investigation has its source in an article by the author: *On the notion of measurability*[1].

1. Let X be an abstract space (an arbitrary set), and μ be an outer measure of Carathéodory on X, i.e., μ is a real valued function $\mu(A)$ defined for each subset A of X satisfying the following conditions:

i) $0 \leqq \mu(A) \leqq +\infty$. ii) If $A_1 \subset A_2$ then $\mu(A_1) \leqq \mu(A_2)$. iii) For any sequence of sets $\{A_n\}$ $(A_n \subset X)$ it holds the relation $\mu(\bigcup_{n=1}^{\infty} A_n) \leqq \sum_{n=1}^{\infty} \mu(A_n)$. iv) $\mu(0) = 0$ for the empty set 0.

We denote by $\mathfrak{C}(\mu)$ *the class of all measurable sets in the sense of Carathéodory with respect to the outer measure* μ[2]. We assume further that *there exists a sequence of sets* $\{K_n\}$ *such that* $K_n \in \mathfrak{C}(\mu)$, $K_n \subset K_{n+1}$, $\bigcup_{n=1}^{\infty} K_n = X$ *and* $\mu(K_n) < +\infty$. We call such a sequence $\{K_n\}$ a *fundamental finite series.* If $\mu(X) < +\infty$, then we can take $K_n = X$.

We say that a class of sets $\mathfrak{M}$ is *completely additive,* when $\mathfrak{M}$ satisfies the following conditions:

a) If $A_i \in \mathfrak{M}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$. *b*) If $A \in \mathfrak{M}$, then $CA \in \mathfrak{M}$[3].

We say that $\mathfrak{M}$ is *finitely additive,* when in *a*) $\bigcup_{i=1}^{\infty}$ is replaced by $\bigcup_{i=1}^{2}$.

We say that $\mathfrak{M}$ is *μ-completely additive* (abbreviated *μ-c.a.*), when $\mathfrak{M}$ is completely additive and the relation $\mu(\bigcup_{i=1}^{\infty} (A_i \frown K_n)) = \sum_{i=1}^{\infty} \mu(A_i \frown K_n)$ (for all n) holds, if $A_i \in \mathfrak{M}$, $A_i \frown A_j = 0$ $(i \neq j)$, and $\{K_n\}$ is a fundamental finite series. We say that $\mathfrak{M}$ is *μ-finit ly additive* (abbreviated *μ-f.a.*), when $\mathfrak{M}$ is finitely additive and the above relation holds if $\bigcup_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}$ are replaced by $\bigcup_{i=1}^{2}$ and $\sum_{i=1}^{2}$ resp.

These definitions are *independent of the choice of the fundamental finite series* $\{K_n\}$ (by Lemma 4), and coincide with the ordinary one if $\mu(\bigcup A_i) < +\infty$ (by Lemma 3).

Let $\mathfrak{N}(\mu)$ be *the class of all sets A such that*

$$\mu(K_n \frown A) = \mu(K_n) - \mu(K_n \frown CA) \qquad \textit{for all } n,$$

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1) By S. Enomoto, this proc. vol. 27, No. 5, p. 208. It will be denoted by [E].

2) A set E is said to be measurable in the sense of Carathéodory when $\mu(A) = \mu(A \frown E) + \mu(A \frown CE)$ holds for all $A \subset X$.

3) CA denotes the compliment of A: $CA = X - A$.