135. On Linear Modulars.

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Let R be a modulared semi-ordered linear space[1] with a modular m. If R is semi-regular, we can introduce into R two sorts of norms, namely, *the first norm* $\|a\|$ $(a\varepsilon R)$ and *the second norm* $|||a|||$ $(a\varepsilon R)$, satisfying the condition

$$|||a||| \leqq \|a\| \leqq 2|||a||| \qquad (a\varepsilon R).$$

It is proved that, if m is linear or singular[2], then we have

$$(*) \qquad \|a\| = |||a||| \qquad (a\varepsilon R).$$

In this paper we will prove the converse, that is:

Theorem. If a modulared semi-ordered linear space R with a modular m is semi-regular and the condition (*) is always satisfied, then m is either linear or singular.

Suppose, in the sequel, that the condition (*) is satisfied and we denote the common value by $\|a\|$ $(a\varepsilon R)$.

Lemma 1. The first norm and the second norm by the conjugate modular $\overline{m}$ of m coincide.

Proof. The first norm by $\overline{m}$ is the conjugate norm of the second norm by m, and the second norm by $\overline{m}$ is the conjugate norm of the first norm by m. Hence our assertion is obtained.

Lemma 2. For a element a such that $\|a\| = 1+m(a)$, we have $m(a)=0$.

Proof. Suppose $m(a) \geqq 1$. Then we have $m(a) \geqq \|a\|$ by the definition of the second norm, contradicting the assumption. Thus we have $m(a) < 1$, and hence $\|a\| \leqq 1$[3]. Therefore, from the assumption, we conclude $m(a)=0$.

Lemma 3. If there is a simple domestic element a satisfying the condition $m(a)=1$, then m is a linear modular on $[a]R$.

Proof. As a is simple and domestic, we can find a positive element $\bar{a}$ of the conjugate space $\overline{R}$ of R such that

$$\bar{a}(a) = \overline{m}(\bar{a}) + m(a), \quad \text{and} \quad [\bar{a}]^R = [a].$$

From this relation, we conclude $\|\bar{a}\| = \bar{a}(a) = \overline{m}(\bar{a})+1$, because, for the first norm $\|\bar{a}\|$ by $\overline{m}$, we have

$$\|\bar{a}\| = \sup_{m(x)\leqq 1} |\bar{a}(x)|, \quad \text{and} \quad |\bar{a}(x)| \leqq \overline{m}(\bar{a}) + m(x) \qquad (x\varepsilon R)$$

Thus we obtain $\overline{m}(\bar{a}) = 0$ by the previous lemma.

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1) H. Nakano: Modulared semi-ordered linear spaces. Tokyo Math. Book Series, Vol. I (1950), p. 153.
2) ibid., p. 184.
3) ibid., p. 181.