75. On Rings of Operators of Infinite Classes. II.

By Haruo SUNOUCHI.

Mathematical Institute, Tôhoku University, Sendai.

(Comm. by Z. SUETUNA, M.J.A., July 12, 1952.)

In the previous paper [5], we have extended the notion of the $\natural$-operation, introduced by Dixmier [1], to the rings of operators of the infinite classes. But the statements of the last section of [5] are not complete, therefore we will precisely discuss them with some modifications. Especially, we shall clarify the relation between the finiteness and the E-finiteness of a projection. By the way, we obtain a functional characterisation of the abelian rings of operators, which is a generalisation of von Neumann's one in separable cases [3; Theorem 6].

1. Firstly we shall remember some definitions. Let $\boldsymbol{M}$ be a ring of operators in a Hilbert space $\boldsymbol{H}$, and denote the center by $\boldsymbol{M}^{\natural}$. A projection $P \in \boldsymbol{M}$ is called *finite* if, for any projection $Q \in \boldsymbol{M}$, $P \sim Q \leqq P$ implies $Q=P$, and *infinite* if this is not the case. If the unit element $I \in \boldsymbol{M}$ is finite, then we say $\boldsymbol{M}$ is of a *finite class*, and otherwise $\boldsymbol{M}$ is of an *infinite class*. As remarked in [5], any ring of operators $\boldsymbol{M}$ is decomposed into the direct sum of three rings of operators, $\boldsymbol{M}^f$, $\boldsymbol{M}^i$, and $\boldsymbol{M}^{pi}$, say; $\boldsymbol{M}^f$ is of the finite class, $\boldsymbol{M}^i$ is the one, in which every central projection is infinite but there exists a finite projection in it, and $\boldsymbol{M}^{pi}$ is in the other case. We say $\boldsymbol{M}^{pi}$ is *of the purely infinite class*. For a while, we shall assume that $\boldsymbol{M}=\boldsymbol{M}^i$, because, in $\boldsymbol{M}^f$, the Dixmier theory is applicable, and in $\boldsymbol{M}^{pi}$, our arguments are not available.

By a *central envelope* of a finite projection E we mean the central projection Z, which is the least upper bound of $F \in \boldsymbol{M}$ equivalent to E. Then there is a system of finite projections $E_\alpha \in \boldsymbol{M}$, such that each E_α has no comparable part to others and the corresponding central envelopes Z_α span the unit I. Denote $E=\sum \oplus E_\alpha$ for this system.

Lemma 1.1. *Let E_α be the finite projections in $\boldsymbol{M}$, which have no comparable parts to each other, then $E=\sum \oplus E_\alpha$ is also finite.*

Proof. The assumption is equivalent to that the corresponding central envelopes Z_α are mutually orthogonal. Let $Z=\sum \oplus Z_\alpha$, then Z is obviously the central envelope of E. Any projection $F \in \boldsymbol{M}_{(Z)}$[1] is written in the form: $F=\sum \oplus F_\alpha$, where $F_\alpha=FZ_\alpha$. Naturally

1) $\boldsymbol{M}_{(E)}$ denotes the set of all $A_{(E)}=EA=AE$, $A \in \boldsymbol{M}$.