36. *Note on S-Groups*

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We want, in the present note, to show that some theorems concerning finite soluble groups can be extended to soluble groups admitting the maximal condition for subgroups, which we call, after K.A. Hirsch, S-groups. The proofs are based on a fundamental theorem of K.A. Hirsch, which states that the structural property of an S-group G occurs in a certain finite factor group of G, and are reduced to those on finite soluble groups.

(0)[1] We can formulate the fundamental theorem of K.A. Hirsch in an a little stronger (though formally) form :

Let G be an S-group. Let H be a normal subgroup of G. Assume that H is neither strongly soluble[2] nor nilpotent nor abelian. Then there exists a family of normal subgroups $\{N_\alpha\}$ of G satisfying the following conditions ; (1) each N_α is contained in H and has the finite index with respect to H, (2) each H/N_α is neither strongly soluble nor nilpotent nor abelian and (3) $\bigcap_\alpha N_\alpha = 1$.

(1)[3] [H. Wielandt] Let G be an S-group. Let $\Phi(G)$ and $D(G)$ be the Frattini subgroup and the commutator subgroup of G respectively. If $\Phi(G)$ contains $D(G)$, then G is nilpotent.

Proof. If G is not nilpotent, G contains a normal subgroup N such that G/N is finite, is not nilpotent and satisfies the same condition as G, which contradicts with a theorem of H. Wielandt.

(2)[*] The Frattini subgroup $\Phi(G)$ of an S-group G is nilpotent.

Proof. We may assume that $\Phi(G)$ is finite. In fact, if $\Phi(G)$ is not nilpotent, then $\Phi(G)$ contains a normal subgroup N such that N is normal also in G and $\Phi(G)/N$ is finite and not nilpotent. So, let us assume that $\Phi(G)$ is finite. Then we can take a normal subgroup H such that $H \cap \Phi = 1$ and G/H is finite. Thus the assertion is reduced to the case of finite groups, for which the assertion is well known.

(3)[4] Let G be a nilpotent S-group. Let H be a subgroup of G such that $D(H)$ is contained properly in $D(G)$. Then $D(\Phi(G)\cdot H)$ is also contained properly in $D(G)$.

Proof. First we assume that the assertion is valid for normal subgroups and that H is not normal in G. Let L be the least normal subgroup of G containing H. If $D(L)\neq D(G)$, then $D(\Phi(G)\cdot L)\neq D(G)$ by assumption. Then $D(\Phi(G)\cdot H)\neq D(G)$, a fortiori. If $D(L)=D(G)$,

*) This theorem has been obtaiend by K.A. Hirsh independently (unpublished).