97. Note on Dirichlet Series. X. Remark on S. Mandelbrojt's Theorem

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(1) **Introduction.** Let us put

(1.1) $F(s)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s=\sigma+it,\ 0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n\to+\infty).$

Let $F(s)$ be uniformly convergent in the whole plane. Then $F(s)$ defines the integral function, and for any given σ, $\operatorname{Sup}_{-\infty<t<+\infty}|F(\sigma+it)|$ has the finite value $M(\sigma)$. After J. Ritt[1] (pp. 18–19), we can define the order and type of $F(s)$ as follows:

Definition I. *The order* σ *of* (1.1) *is defined by*

$$\rho=\overline{\lim_{\sigma\to-\infty}}\ 1/(-\sigma).\ \log^+\log^+ M(\sigma),$$

where $\log^+ x = Max(0, \log x)$. *If* $0<\rho<+\infty$, *then the type* k *of* (1.1) *is defined by*

$$k=\overline{\lim_{\sigma\to-\infty}}\ 1/\exp((-\sigma)\rho).\ \log^+ M(\sigma).$$

Definition II. *Let* $D(r;C)$ *be the curved strip which is generated by circles with radii* r, *and having its centres on the analytic curve* C, *which extends to* $\Re(s)=-\infty$. *Then the order* $\rho(D)$ *in* D *is defined by*

$$\rho(D)=\lim 1/(-\sigma).\ \log^+\log^+ M(\sigma;D),$$

where $M(\sigma;D)=\operatorname{Max}_{s\in D,\ \Re(s)=\sigma}|F(s)|$. *If* $0<\rho(D)<+\infty$, *then the type* $k(D)$ *in* D *is defined by*

$$k(D)=\overline{\lim_{\sigma\to-\infty}}\ 1/\exp((-\sigma)\rho(D)).\ \log^+ M(\sigma;D).$$

S. Mandelbrojt has proved the following.

Theorem (S. Mandelbrojt[1] p. 19). *Let* (1.1) *with* $\underline{\lim}_{n\to+\infty}(\lambda_{n+1}-\lambda_n)=h>0$, $\overline{\lim}_{n\to+\infty} n/\lambda_n=\delta(\leqq 1/h)$ *be simply (necessarily absolutely) convergent in the whole plane. Then, in any strip:* $|\Im(s)-t|\leqq\pi(\delta+\varepsilon)$ (t: *arbitrary but fixed,* ε: *any given positive constant*), (1.1) *has the same order as in the whole plane.*

In this note, we shall generalize it as follows:

Theorem. *Let* (1.1) *with* $\underline{\lim}_{n\to+\infty}(\lambda_{n+1}-\lambda_n)=h>0$, $\overline{\lim}_{n\to+\infty} n/\lambda_n=\delta$ ($\leqq 1/h$) *be simply (necessarily absolutely) convergent in the whole plane. Then, in any curved strip* $D(\pi(\delta+\varepsilon);C)$ (ε: *any given positive constant*), (1.1) *has the same order as in the whole plane.*

If furthermore $\delta=0$, *then in* $D(\varepsilon;C)$, (1.1) *has the same order and type as in the whole plane.*