113. On the Transformations Preserving the Canonical Form of the Equations of Motion

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Introduction. In this paper, we shall prove that any transformation preserving the canonical form of the equations of motion can be composed of a canonical transformation and a transformation of the form $Q_i = \rho q_i$, $P_i = p_i$ $i=1,\dots,n$ where $\rho \neq 0$ is a constant. (For the precise formulation, see section 3, 4.)

For the sake of completeness, we shall prove first some lemmas on matrices which will be used later.

1. We shall call a real regular matrix A of degree $2n$, *a real quasi-symplectic matrix* (we abbreviate it as r.q.s.m.) *with a multiplier* ρ, if

$$\rho \sum_{i=1}^{n} (x_i y_{i+n} - x_{i+n} y_i) = \sum_{i=1}^{n} (x_i' y_{i+n}' - x_{i+n}' y_i') \tag{1}$$

for two arbitrary vectors $(x_1,\dots,x_{2n})$, $(y_1,\dots,y_{2n})$, where ρ is a real number and

$$\begin{pmatrix} x_1' \\ \vdots \\ x_{2n}' \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_{2n} \end{pmatrix} \qquad \begin{pmatrix} y_1' \\ \vdots \\ y_{2n}' \end{pmatrix} = A \begin{pmatrix} y_1 \\ \vdots \\ y_{2n} \end{pmatrix}.$$

A r.q.s.m. with the multiplier 1 is called *a real symplectic matrix* (we abbreviate it as r.s.m.). A real regular matrix A of degree $2n$ is a r.q.s.m. with a multiplier ρ if and only if

$$\rho J = A^* J A \tag{2}$$

where A^* is the transposed of A and

$$J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} \quad (E_n \text{ is the unit matrix of degree } n).$$

From (2), a multiplier of a r.q.s.m. is a non-vanishing real number.

A real matrix B of degree $2n$ is called *an infinitesimal real symplectic matrix* (we abbreviate it as i.r.s.m.), if

$$JB + B^* J = 0. \tag{3}$$

If we write a real matrix B of degree $2n$ in the form

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where B_1, B_2, B_3, B_4 are matrices of degree n, then B is an i.r.s.m. if and only if

$$B_4 = -B_1^*, \qquad B_3 = B_3^*, \qquad B_2 = B_2^*. \tag{4}$$

2. **Lemma 1.** *Let $A(t)$, $B(t)$ be real matrices of degree $2n$ de-*