109. Note on Dirichlet Series. XI. On the Analogy between Singularities and Order-curves

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(1) **Introduction.** Let us put

(1.1) $F(s)=\sum_{n=1}^{\infty}a_n\exp(-\lambda_n s)\quad (s=\sigma+it,\ 0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n\to+\infty)$.

O. Szász has proved the next theorem, which is a generalization of Hurwitz-Pólya's theorem (E. Landau[1], p. 36).

O. Szász's Theorem (O. Szász[2], p. 107). *Let* (1.1) *have the finite simple convergence-abscissa* σ_s. *If* $\lim_{n\to+\infty}\log n/\lambda_n=0$, *then there exists a sequence* $\{\varepsilon_n\}$ $(\varepsilon_n=\pm1)$ *such that* $\sum_{n=1}^{\infty}a_n\varepsilon_n\exp(-\lambda_n s)$ *has* $\sigma=\sigma_s$ *as the natural boundary.*

The author proved recently the following theorem of the same type:

Theorem (C. Tanaka[3], p. 308). *Let* (1.1) *have the finite simple convergence-abscissa* σ_s. *If* $\lim_{n\to+\infty}\log n/\lambda_n=0$, *then there exists a Dirichlet series* $\sum_{n=1}^{\infty}b_n\exp(-\lambda_n s)$ *having* $\sigma=\sigma_s$ *as the natural boundary such that*

(a) $|b_n|=|a_n|$ $(n=1,2,\dots)$ *and* $\lim_{n\to+\infty}|\arg(a_n)-\arg(b_n)|=0$

or

(b) $\arg(b_n)=\arg(a_n)$ $(n=1,2,\dots)$ *and* $\lim_{n\to+\infty}|b_n/a_n|=1$.

In this note, we shall establish analogous theorems concerning order-curves. We first begin with

Definition. *Let* (1.1) *be uniformly convergent in the whole plane. Then, we call the analytic curve* C *extending to* $\sigma=-\infty$ *the order-curve of* (1.1), *provided that, in* $D(\varepsilon\,;C)$ (ε*: any positive constant*), (1.1) *has the same order as in the whole plane, where* $D(\varepsilon\,;C)$ *is the curved strip generated by circles with radii* ε *and having its centres on* C.

Our theorems read as follows:

Theorem I. *Let* (1.1) *with* $\overline{\lim_{n\to+\infty}}\log n/\lambda_n<+\infty$ *be simply* (*necessarily absolutely*) *convergent in the whole plane, and* C *be any given analytic curve extending to* $\sigma=-\infty$. *Then, there exists a everywhere absolutely convergent Dirichlet series* $\sum_{n=1}^{\infty}\varepsilon_n a_n\exp(-\lambda_n s)$ $(\varepsilon_n=\pm1)$, *such that it has every curve* C_τ $(-\infty<\tau<+\infty)$ *as its order-curve, where* C_τ *is obtained from moving* C *in parallel by* $i\tau$ $(-\infty<\tau<+\infty)$.

Theorem II. *Under the same assumptions as above, there exists*