125. Notes on Some Theorems on the Sphere

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Borsuk[2] proved that if f is a continuous mapping of the n-dimensional sphere S^n into the n-dimensional Euclidean space E^n, then f maps some pair of antipodal points into a single point, which had been conjectured by Ulam. This Borsuk-Ulam theorem has been extended by Tucker[4] such that if f is a continuous mapping of S^n into itself with the degree 0, then f maps some pair of antipodal points into a single point. In this note in §1 we shall have an extension of these theorems.

Borsuk[2] proved also that if S^n is covered by $n+1$ closed sets, then at least one of them contains an antipodal pair, which is now called the theorem of Lusternik-Schnirelmann-Borsuk. In §2 we shall have an extension of this theorem and a consequence of this extension.

§1. Now we prove the following:

Theorem 1. *Let f be a continuous mapping of S^n into itself. If f has an even degree, then f maps some pair of antipodal points into a single point.*

Proof. Assume that S^n is the unit sphere in E^{n+1}. Let f be a continuous mapping which satisfies the condition of Theorem. Suppose on the contrary that $f(x) \neq f(x^*)$ for every $x \in S^n$, where x^* is the antipodal point of x. Using vectorial notation, put

$$g(x) = \frac{f(x)-f(x^*)}{|f(x)-f(x^*)|}.$$

Then g is a continuous mapping of S^n into itself. Since

$$g(x^*) = \frac{f(x^*)-f(x)}{|f(x^*)-f(x)|} = -g(x)$$

for every $x \in S^n$, g maps antipodal points of S^n into antipodal points of S^n. Therefore g has an odd degree by a theorem of Borsuk[2].

Now we shall prove that $|f(x)-g(x)|<2$ for every $x \in S^n$. Since from this fact it follows that g will be homotopic to f and that g will have the same degree to that of f (i.e. an even degree), we shall have a contradiction, and the proof of Theorem will be complete.

To prove that $|f(x)-g(x)|<2$ for every $x \in S^n$, suppose on the contrary that there exists a point $p \in S^n$ with $|f(p)-g(p)|=2$. Then we have

$$f(p) = -g(p) = -\frac{f(p)-f(p^*)}{|f(p)-f(p^*)|}.$$