124. On the Existence of Periodic Solutions for Certain Differential Equations

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In this note we shall give the existence theorems on the periodic solutions of the differential equations

$$(1) \qquad \frac{d}{dt}\left(a(x)\frac{dx}{dt}\right)+f(x)\frac{dx}{dt}+g(x)=e(t)$$

$$(2) \qquad a(x)\frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=e(t)$$

where $e(t)$ is a periodic function of t with the least positive period ω and $\int_0^\omega e(t)dt=0$, and $|e(t)|\leqq e$. Moreover, we suppose that $a'(x)$, $g(x)$ and $e(t)$ have continuous derivatives and $f(x)$ is a continuous function.

Of course, the proofs of the following theorems follow from the fixed point theorem. Therefore, it is sufficient to show that the existence of a curve which encloses the domain satisfying the hypotheses of the fixed point theorem.

Theorem 1. *Suppose that the following conditions are satisfied:*

(a) $a(x)>0$ *for all* x.

(b) $\int_0^x f(x)\,dx\,(=F(x))\to\pm\infty$ *as* $x\to\pm\infty$ *respectively.*

(c) *There exists a positive number* x_0 *such that* $x\cdot g(x)\geqq 0$ *for* $|x|\geqq x_0$.

Then the equation (1) *has at least one periodic solution of period* ω.

Proof. We consider a pair of first order equations,

$$(3) \qquad \begin{cases} a(x)\dfrac{dx}{dt}=y-F(x)+E(t)=y-F(x)+\displaystyle\int_0^t e(t)\,dt \\ \dfrac{dy}{dt}=-g(x) \end{cases}$$

instead of the equation (1).

For a positive number ε, we choose an x-value $\xi\,(\geqq x_0)$ such that

$$F(x)>\max_t E(t)+\varepsilon \qquad \text{for } x\geqq\xi,$$

$$F(x)<\min_t E(t)-\varepsilon \qquad \text{for } x\leqq-\xi,$$

and a positive number η such that $\eta\leqq\varepsilon/A(\xi)$ and $\eta\leqq-\varepsilon/A(-\xi)$ where $A(x)=\int_0^x a(x)\,dx$.

Now, we consider three functions