123. On Spaces Having the Weak Topology with Respect to Closed Coverings

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Let X be a topological space and $\{A_\alpha\}$ a closed covering of X. We shall say that *X has the weak topology with respect to $\{A_\alpha\}$*, if the union of any subcollection $\{A_\beta\}$ of $\{A_\alpha\}$ is closed in X and any subset of $\bigcup_\beta A_\beta$ whose intersection with each A_β is open relative to the subspace topology of A_β is necessarily open in the subspace $\bigcup_\beta A_\beta$; the word "open" may, of course, be replaced by "closed".

According to this definition any CW-complex K in the sense of J. H. C. Whitehead[1] has the weak topology with respect to the closed covering which consists of the closures of all the cells of K. Thus the theorems concerning spaces having the weak topology with respect to closed coverings are applicable to CW-complexes which play an important rôle in algebraic topology.

Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}$. In this paper we are concerned primarily with the problem: what property of each subspace A_α has influence upon the whole space X? For example, if each A_α consists of a single point (or more generally if each A_α is discrete), X is discrete. It will be shown below that if each subspace A_α is (completely or perfectly) normal, so is X. Our main theorem is that if each subspace A_α is metrizable, then any subset of X is paracompact and perfectly normal. Since the closure of each cell of a CW-complex is a compact metrizable space, it follows immediately from our theorem that any subset of a CW-complex is paracompact and perfectly normal[2].

§ 1. Product Spaces.

Lemma 1. *Let $\{A_\alpha\}$ be a locally finite (=neighbourhood finite in the sense of S. Lefschetz) closed covering of a topological space X. Then X has the weak topology with respect to $\{A_\alpha\}$.*

Lemma 2. *Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}$. Then a mapping f of X into*

1) J. H. C. Whitehead, Bull. Amer. Math. Soc., **55**, 213-245 (1949).

2) The paracompactness is proved independently for simplicial complexes with the weak topology by D. G. Bourgin, Proc. Nat. Acad. Sci. U.S.A., **38**, 305-313 (1952); J. Dugundji, Portugaliae Math., **11**, 7-10-b (1952); H. Miyazaki, Tohoku Math. Jour., **4**, 83-92 (1952); K. Morita, Amer. Jour. Math., **75**, 205-223 (1953) and for CW-complexes by H. Miyazaki, Tohoku Math. Jour., **4**, 309-313 (1952).