193. *Unitary Representations of SL(n, C)*

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§1. Let G be the group $SL(n, C)$ and $\chi=\{\lambda_1, \mu_1; \cdots; \lambda_{n-1}, \mu_{n-1}\}$ (λ_i and μ_i are complex numbers and $\lambda_i-\mu_i$ are integers) be a character of the diagonal subgroup $D: \chi(\delta)=(\delta_2\delta_3\cdots\delta_n)^{(\lambda_1,\mu_1)}(\delta_3\cdots\delta_n)^{(\lambda_2,\mu_2)}\cdots\cdots\delta_n^{(\lambda_{n-1},\mu_{n-1})}$ (where $z^{(\lambda,\mu)}=z^{\lambda}\bar{z}^{\mu}$), and W be the Weyl group G whose elements can be identified with $s=\begin{pmatrix}1\cdots n\\ j_1\cdots j_n\end{pmatrix}$ of the permutation group $\mathfrak{S}_n$ generated by $s_i=(i\ i+1)$. We divide the set of characters into two classes: regular and singular. A regular character χ is such that any one of pairs (λ_i', μ_i') contained in $\chi'=\chi^s$ for all $s\in W$ is not a pair of integers of the same signature, and a singular character χ is otherwise. Furthermore we call the singular character χ to be of type (D), if some pair in χ is $(-1, -1)$ and any other pair (λ_i', μ_i') in $\chi'=\chi^s$ for s, such that s leaves the totality of pairs $(-1, -1)$ in χ stable, is not a pair of integers of the same signature.

In this paper we shall discuss the unitarity of the elementary representation $R(\chi)=\{T^{\chi}, \mathcal{D}_{\chi}\}$ of G for a regular character χ and a singular character χ of type (D). We shall remark that the unitary representation of the degenerate supplementary series recently described by E. M. Stein [1] is one of the representations $R(\chi)$ of the latter case.

§2. Generalizing the method in [2] and [3], we consider the invariant bilinear form between two representations $R(\chi)$ and $R(\chi')$. In this point of view the integral kernel of the invariant bilinear form is analytically continued rather than the representation itself.

Let $\langle\varphi, \psi\rangle=B(\varphi, \bar{\psi})$ for φ and $\psi\in\mathcal{D}_{\chi}$, where $B(\cdot, \cdot)$ is an invariant bilinear form on $\mathcal{D}_{\chi}\times\mathcal{D}_{\bar{\chi}}$, and then $\langle\cdot, \cdot\rangle$ is an Hermitian form on $\mathcal{D}_{\chi}$. If $\langle\cdot, \cdot\rangle$ exists and is positive definite, the representation $R(\chi)$ is unitary with respect to this scalar product. In order that the invariant Hermitian form on $\mathcal{D}_{\chi}$ exists, it is necessary and sufficient that χ satisfied the condition that $\chi\bar{\chi}^s=1$ for some $s\in W$.

§3. Let χ be a regular character satisfying $\chi\bar{\chi}^s=1$ for some $s\in W$, then we have a non-degenerate Hermitian form on $\mathcal{D}_{\chi}$. Now if we set $\chi'=\chi^{s_i}$, then the representations $R(\chi)$ and $R(\chi')$ are equivalent by means of the intertwinning operator A_i:

$$A_i\varphi(z)=\gamma(\lambda_i, \mu_i)\int z'^{(\lambda_i-1,\mu_i-1)}_{i+1\,i}\varphi(z'_{i+1\,i}z)dz'_{i+1\,i}.$$