139. *On Radicals of Semigroups with Zero. I*

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The term "semigroup" means in this note always a semigroup with zero element (see [3]). Several concrete types of radicals for semigroups were proposed (see for instance [2], [3], [5]–[9] and [11]). By a ring theoretical analogy (see [4]) also a general theory of radicals for semigroups can be developed.

For any class $\boldsymbol{C}$ of semigroups a $\boldsymbol{C}$-semigroup S means a semigroup belonging to $\boldsymbol{C}$. If a semigroup S has a $\boldsymbol{C}$-ideal $\boldsymbol{C}(S)$ such that $\boldsymbol{C}(S)$ contains any further $\boldsymbol{C}$-ideal of S, then $\boldsymbol{C}(S)$ is called the $\boldsymbol{C}$-radical of S. Semigroups S with $\boldsymbol{C}(S)=0$ are called $\boldsymbol{C}$-semisimple. A class R of semigroups is called *radical*, if the following conditions are satisfied:

1) $\boldsymbol{R}$ is homomorphically closed/not only with respect to forming of Rees factor semigroups/

2) in any semigroup S there exists the $\boldsymbol{R}$-radical $\boldsymbol{R}(S)$

3) the Rees factor semigroup $S/\boldsymbol{R}(S)$ is $\boldsymbol{R}$-semisimple.

The aim of this note is to generalize for semigroups some ring-theoretical results of [1] and [10].

Theorem 1. *For any radical class $\boldsymbol{R}$ of semigroups, and for any ideal $\boldsymbol{J}$ of a semigroup S, the $\boldsymbol{R}$-radical $\boldsymbol{R}(\boldsymbol{J})$ of $\boldsymbol{J}$ is an ideal of S.*

Proof. Assuming that $\boldsymbol{R}(\boldsymbol{J})$ is not an ideal of S, there exists an element $s\in S$ satisfying either $s\boldsymbol{R}(\boldsymbol{J})\not\subseteq\boldsymbol{R}(\boldsymbol{J})$ or $\boldsymbol{R}(\boldsymbol{J})s\not\subseteq\boldsymbol{R}(\boldsymbol{J})$. If $s\boldsymbol{R}(\boldsymbol{J})\not\subseteq\boldsymbol{R}(\boldsymbol{J})$, then the union $\boldsymbol{U}=s\boldsymbol{R}(\boldsymbol{J})\cup\boldsymbol{R}(\boldsymbol{J})$ properly contains $\boldsymbol{R}(\boldsymbol{J})$ and $\boldsymbol{U}\subseteq\boldsymbol{J}$ holds. By $\boldsymbol{J}\boldsymbol{U}=\boldsymbol{J}s\boldsymbol{R}(\boldsymbol{J})\cup\boldsymbol{J}\boldsymbol{R}(\boldsymbol{J})\subseteq\boldsymbol{R}(\boldsymbol{J})$ and $\boldsymbol{U}\boldsymbol{J}\subseteq\boldsymbol{U}$ this union $\boldsymbol{U}$ is an ideal of $\boldsymbol{J}$. Being $\boldsymbol{J}/\boldsymbol{R}(\boldsymbol{J})$ $\boldsymbol{R}$-semisimple, $\boldsymbol{U}/\boldsymbol{R}(\boldsymbol{J})$ is not an $\boldsymbol{R}$-semigroup.

By $\varphi_1(r)=sr\cup\boldsymbol{R}(\boldsymbol{J})$ $(r\in\boldsymbol{R}(\boldsymbol{J}))$ is given a mapping of $\boldsymbol{R}(\boldsymbol{J})$ onto $\boldsymbol{U}/\boldsymbol{R}(\boldsymbol{J})$, which by the associativity and

$$\begin{aligned}\varphi_1(r_1, r_2)&=sr_1r_2\cup\boldsymbol{R}(\boldsymbol{J})=\boldsymbol{R}(\boldsymbol{J})\\&=sr_1s, r_2\cup\boldsymbol{R}(\boldsymbol{J})=\varphi_1(r_1), \varphi_1(r_2)\end{aligned}$$

is a homomorphism. Being $\boldsymbol{R}(\boldsymbol{J})$ radical and $\boldsymbol{U}/\boldsymbol{R}(\boldsymbol{J})$ nonradical nonzero semigroups, respectively, this contradiction shows $S\boldsymbol{R}(\boldsymbol{J})\subseteq\boldsymbol{R}(\boldsymbol{J})$. Similarly can be verified also $\boldsymbol{R}(\boldsymbol{J})S\subseteq\boldsymbol{R}(\boldsymbol{J})$.

Corollary 2. *With the above notations $\boldsymbol{R}(\boldsymbol{J})\subseteq\boldsymbol{J}\cap\boldsymbol{R}(S)$ holds.*

Proof. $\boldsymbol{R}(\boldsymbol{J})$ is an $\boldsymbol{R}$-ideal of S, contained in $\boldsymbol{R}(S)$.

Corollary 3. *Any ideal of an $\boldsymbol{R}$-semisimple semigroup is again*