175. *On a Theorem of Koebe for Quasiconformal Mappings*

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1. Let $w=f(z)$ be a quasiconformal mapping, whose dilatation quotient is bounded above by $K(\geqq 1)$, of the unit disc $|z|<1$ into a domain in $|w|<\infty$ in the sense of Grötzsch. We call such mapping a K-quasiconformal mapping of $|z|<1$ into $|w|<\infty$. If $w=f(z)$ is a K-quasiconformal mapping for some K, then it is called a quasiconformal mapping. We denote by S_α and $S_\alpha(K)$, respectively, the families of quasiconformal mappings and K-quasiconformal mappings in Grötzsch sense such that each mapping $f(z)$ is univalent in $|z|<1$ and $f(0)=0$ and $\lim_{z\to 0}|f(z)|/|z|^\alpha=1$, where α is real.

We denote by $\mathfrak{S}_\alpha$ and $\mathfrak{S}_\alpha(K)$, respectively, the families of quasiconformal mappings and K-quasiconformal ones of $|z|<1$ into a domain in $|w|<\infty$ in the sense of Pfluger-Ahlfors such that these mappings satisfy the same normalization as above at the origin. A univalent quasiconformal mapping in Grötzsch sense is a continuously differentiable quasiconformal one in Pfluger-Ahlfors sense. Then we have

$$S_\alpha\subset\mathfrak{S}_\alpha,\ S_\alpha(K)\subset\mathfrak{S}_\alpha(K)\quad\text{and}\quad\mathfrak{S}_\alpha(K)\subset\mathfrak{S}_\alpha.$$

2. Y. Juve [2] extended Koebe's quarter-disc theorem to the family $S_{1/d(0)}$, where $d(0)$ means the value at the origin of the dilatation quotient of $w=f(z)$, and proved the following theorem:

Theorem. *Let $w=f(z)$ be any mapping belonging to $S_{1/d(0)}$. Denote by δ the distance from the origin to the boundary of the image domain of $|z|<1$ under $w=f(z)$. Then*

$$\delta\geqq\frac{1}{4}\exp\left\{-\int_0^1\left(\frac{1}{d(0)}-\frac{1}{\frac{1}{2\pi}\int_0^{2\pi}d(z)d\theta}\right)\frac{dr}{r}\right\}.$$

Here, an extremal mapping giving the equality is the composite mapping of

$$\zeta=|z|^{1/d(0)}\{1+(R-1)|z|^{R/(1-R)d(0)}\}e^{i\arg z}$$

and $w=\zeta\Big/\left(1+\dfrac{\zeta}{R}\right)^2$, where

$$R=\exp\left\{-\int_0^1\left(\frac{1}{d(0)}-\frac{1}{\frac{1}{2\pi}\int_0^{2\pi}d(z)d\theta}\right)\frac{dr}{r}\right\}.$$

3. The proof for this theorem by Juve [2] is a modification of that for Ahlfors' distortion theorem (cf. R. Nevanlinna [3], Kap. IV,