174. *Structure of Maximal Sum-free Sets in Groups of Order 3p*

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(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1970)

1. Introduction. In [5] and [6], we studied the structure of maximal sum-free sets of elements in groups of prime orders $p=3k+2$ and $p=3k+1$ respectively. In this paper, we shall study the structure of maximal sum-free sets in groups G (both abelian and non-abelian) of order $3p$, where $p=3k+1$ is a prime. We shall use the same terminologies and notations as used in [1]. In particular, we let S be a maximal sum-free set in G and $|S|$ be the cardinal of S.

2. Abelian groups. Throughout this section G is abelian. We first prove that $|S+S|\neq 2|S|$ in Theorem 4 of [1]. In fact, we shall prove

Lemma 1. *If S is a maximal sum-free set in G, then S is a union of cosets of some subgroup H, of order p or 1, such that*

$$|S+S|=2|S|-|H|.$$

Proof. Write $G=\{0,1,2,\cdots,3p-1\}$. Let $H_0=H=\{0,3,6,\cdots,3(p-1)\}$, $H_1=p+H$, $H_2=2p+H$, $S_i=S\cap H_i$, $i=0,1,2$.

If $S=H_1$, say, then it is clear that $|S+S|\neq 2|S|$.

Assume now that $S\neq H_1$ and $S_1\neq\varnothing$. By Theorem 5 of [1], $|S_0|\leqq k$. Thus $|S_1|+|S_2|\geqq 2k+1$ and without loss of generality, we may assume that $|S_1|\geqq k+1$.

Now $(S_1+S_1)\cap S_2=\varnothing$ and $(S_1+S_1)\cup S_2\subseteqq H_2$. Hence, by Cauchy-Davenport theorem ([2], p. 3), if $S_1+S_1\neq H_2$,

$$\begin{aligned} p&\geqq |S_2|+|S_1+S_1|\geqq |S_2|+2|S_1|-1\\ &\geqq k+|S_1|+|S_2|\geqq |S_0|+|S_1|+|S_2|=p, \end{aligned}$$

from which it follows that

$$|S_0|=k,\quad |S_1|=k+1,\quad \text{and}\quad |S_2|=k.$$

(If $S_1+S_1=H_2$, then we can prove that $S_0=\varnothing$ and so $S=H_1$, which contradicts the assumption.)

Let $S^*=-S\cup S$. Then $S^*\neq S$. But from Theorem 4 of [1], we have (i) $|S+S|=2|S|-1$ or (ii) $|S+S|=2|S|$ and $S\cup(S+S)=G$. Thus from $S^*\cap(S-S)=\varnothing$ it follows that $|S+S|\neq 2|S|$.

Hence, in any case $|S+S|\neq 2|S|$.

The proof of Lemma 1 is complete.

Next, we prove

Theorem 1. *Let S be a maximal sum-free set in G such that S is*