254. *Ergodic Properties of Piecewise Linear Transformations*

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1. Introduction. After the work of Rényi [1], ergodic properties of β-expansions of real numbers have been studied in [2]–[4]. In this paper we generalize these results for a class of expansions, called piecewise linear expansions, which includes β-expansions as special cases.

Let $\bar{\beta}=(\beta_0, \beta_1, \cdots, \beta_N), N\geqq 1$, be a $(N+1)$-tuple of positive number such that $0<\theta\equiv\beta_N(1-\sum_{k=0}^{N-1}(1/\beta_R))\leqq 1$.

We denote the set of all $(N+1)$-tuples by $V(N+1)$. For each $\bar{\beta}\in V(N+1)$, we define a corresponding function $f(t)$ by

$$f(t)=\begin{cases}\dfrac{t}{\beta_0}, & 0\leqq t\leqq 1,\\ f(K)+\dfrac{t-k}{\beta_k}, & \begin{array}{l}k<t\leqq k+1,\ (k=1,2,\cdots,N+1),\\ N<t\leqq N+\theta,\ (k=N),\end{array}\\ 1, & t>N+\theta.\end{cases}$$

The function $f(t)$ satisfies the Rényi's conditions [1]. Thus every real number x has the f-expansion

$$x=a_0(x)+f(a_1(x)+f(a_2(x)+\cdots)\cdots),$$

where the digits $a_n(x), n=0,1,\cdots$, and the remainders

$$T^n x=f(a_n(x)+f(a_{n+1}(x)+\cdots)\cdots),\qquad n=0,1,\cdots,$$

are defined by the following recursive relations: $a_0(x)=[x], T^0x=\{x\}$, $T^{n+1}x=\{f^{-1}(T^nx)\}, a_{n+1}(x)=[f^{-1}(T^nx)]$, $n=0,1,\cdots$, where $[z]$ denotes the integral part and $\{z\}$ the fractional part of the real number z and f^{-1} is the inverse function of f.

This f-expansion is called a *piecewise linear expansion induced by $\bar{\beta}$ or simply $\bar{\beta}$-expansion*, and the transformation $Tx=\{f^{-1}(x)\}$, $0\leqq x<1$, is called a *piecewise linear transformation induced by $\bar{\beta}$*. By definition, T is a many to one transformation of $[0,1)$ onto itself and nonsingular with respect to the Lebesgue measure m.

For the number 1, we define, especially, $a_0(1)=0$ and $T^0 1=1$. Then $\bar{\beta}\in V(N+1)$ is said to be *periodic* if the $\bar{\beta}$-expansion of 1 has a recurrent tail, and *rational* if the $\bar{\beta}$-expansion of 1 has a zero tail. The *order of a rational $\bar{\beta}$* is the minimum integer r such that $a_n(1)=0$ for all $n>r+1$.

2. Invariant measures. Lemma 1. *Let T be a piecewise linear transformation induced by $\bar{\beta}\in V(N+1)$ and μ a finite measure equivalent to the Lebesgue measure m. Then μ is T-invariant if and only if*