200. The Multipliers for Vanishing Algebras

By Tetsuhiro SHIMIZU
Department of Mathematics, Tokyo Institute of Technology

(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1971)

Let G be a locally compact Abelian group with Haar measure m. Let Γ be the dual group of G. We denote by $L^1(G)$ the group algebra of G. For any measurable subset S of G, define $L(S)$ to be the subspace of $L^1(G)$ consisting of all functions which vanish locally almost everywhere on the complement of S. When $L(S)$ forms a subalgebra of $L^1(G)$, we call it a vanishing algebra. If $L(S)$ is a vanishing algebra, then we may assume S is a measurable semigroup [2]. In this paper we shall assume $L(S)\neq\{0\}$ to avoid triviality. Let $M(G)$ be the Banach algebra consisting of all bounded regular Borel measures on G. For any Borel set A, put $M(A)=\{\mu\in M(G): \mu \text{ is concentrated on } A\}$.

If A is a Banach algebra, then a mapping $T: A\to A$ is called a multiplier of A if $x(Ty)=(Tx)y(x, y\in A)$.

In this short note, we shall show the characterization of the multipliers for certain vanishing algebras.

Theorem. *If S is an open semigroup, then the space $\mathfrak{M}$ of all multipliers for $L(S)$ is $M(S_0)$, where $S_0=\{t\in G: S\supset S+t$ l.a.e.*)$\}$.*

Proof. At first, we shall show that for any $T\in\mathfrak{M}$ there is a measure $\lambda\in M(G)$ such that $Tf=\lambda*f$ for each $f\in L(S)$ and $\|T\|=\|\lambda\|$. For each f, $g\in L(S)$ we have $(\widehat{Tf})\hat{g}=\hat{f}(\widehat{Tg})$. Since $L(S)$ is contained in no proper colsed ideal of $L^1(G)$ [3], for each $\gamma\in\Gamma$ we can choose a function $g\in L(S)$ such that $\hat{g}(\gamma)\neq 0$. Define $\varphi(\gamma)=(\widehat{Tg})(\gamma)/\hat{g}(\gamma)$. The equation $(\widehat{Tf})\hat{g}=\hat{f}(\widehat{Tg})$ shows that the definition of φ is independent of the choice of g. For φ so defined it is apparent that $(\widehat{Tf})=\varphi\hat{f}$. Let ψ be a second function on Γ such that $(\widehat{Tf})=\psi\hat{f}$ for each $f\in L(S)$. Then since for each $\gamma\in\Gamma$ there is a function $g\in L(S)$ such that $\hat{g}(\gamma)\neq 0$, the equation $(\varphi-\psi)\hat{f}=0$ for each $f\in L(S)$ reveals that $\varphi=\psi$. Evidently, φ is continuous. Let $\gamma_1,\cdots,\gamma_n\in\Gamma$ and $a_1,\cdots,a_n$ be any complex numbers. Let t_0 be a point of S. If $\{x_\alpha\}$ is an approximate identity of $L^1(G)$, then we can assume $(x_\alpha)_{t_0}\in L(S)$, where $(x_\alpha)_{t_0}(t)=x_\alpha(t+t_0)$. Put $b_i=a_i(t_0,\gamma_i)(i=1,2,\cdots,n)$ and $y_\alpha=T((x_\alpha)_{t_0})$. We have that

*) By $A\supset B$ l.a.e., we mean that $B\backslash A$ is locally negligible.