198. *Continuity and Modularity of the Lattice of Closed Subspaces of a Locally Convex Space*

By Shûichirô MAEDA
Ehime University, Matsuyama

(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1971)

1. Introduction. The set of all subspaces of a vector space forms an upper continuous modular atomistic lattice, ordered by set-inclusion. While the set $L_c(E)$ of all closed subspaces of a locally convex space E forms a complete DAC-lattice which is, in general, neither upper continuous nor modular (cf. [3], Chapter VII). The main purpose of this paper is to find some conditions on E under each of which $L_c(E)$ becomes upper continuous, lower continuous and modular respectively. Our main results are as follows: (1) $L_c(E)$ is upper continuous if and only if every subspace of E is closed, (2) $L_c(E)$ is lower continuous if and only if E is a minimal space, (3) in case E is metrisable, $L_c(E)$ is modular if and only if E is a minimal space. The last result is a generalization of a theorem in Mackey [2].

2. Continuity and modularity in DAC-lattices. A lattice L is called *upper continuous* when $a_\delta \uparrow a$ implies $a_\delta \wedge b \uparrow a \wedge b$ and called *lower continuous* when $a_\delta \downarrow a$ implies $a_\delta \vee b \downarrow a \vee b$ ([3], Definition 2.14). We write $(a, b)M$ (resp. $(a, b)M^*$) when the pair (a, b) is modular (resp. dual-modular)([3], Definition 1.1).

Lemma 1. *Let a be an element of a complete lattice L. If the interval $L[a, 1]=\{x \in L; a \leqq x \leqq 1\}$ is upper continuous then for any $b \in L$ there exists a maximal element b_1 such that $b_1 \leqq b$ and $(b_1, a)M^*$.*

An atomistic lattice L with the covering property is called an *AC-lattice* ([3], Definition 8.7). A lattice L with 0 and 1 is called a *DAC-lattice* when both L and its dual are AC-lattices ([3], Definition 27.1). In a DAC-lattice, $(a, b)M$ and $(b, a)M$ are equivalent and so are $(a, b)M^*$ and $(b, a)M^*$ ([3], Theorem 27.6).

Theorem 1. *Let L be a complete DAC-lattice and let $a \in L$. If either $L[a, 1]$ is upper continuous or $L[0, a]$ is lower continuous, then $(a, x)M$ and $(a, x)M^*$ hold for every $x \in L$.*

Corollary. *If a complete DAC-lattice L is either upper or lower continuous then L is modular.*

Lemma 2. *Let L be a complete AC-lattice and assume that there exists a sequence of atoms p_n such that $1=\vee(p_n; 1 \leqq n < \infty)$. If L is sequentially upper continuous ($a_n \uparrow a$ implies $a_n \wedge b \uparrow a \wedge b$) then it is upper continuous.*