196. The Theory of Nuclear Spaces Treated by the Method of Ranked Space. IV

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§ 5. The completion of the linear ranked space Φ, (2).

Lemma 20. *Let $\hat{\Phi}_0$ be the subset of $\hat{\Phi}$ consisting of those equivalence classes which contain an R-Cauchy sequence $\{g_n\}$ for which $g_1=g_2=g_3=\cdots$.*

The mapping T of Φ onto $\hat{\Phi}_0$, which maps $g\in\Phi$ to the class $\hat{g}$ containing the sequence consisting of a single element g, is bijective and we have $g\in V_i\,(0,r,m)$ if and only if $\hat{g}\in\hat{V}_i\,(0,r,m)$.

Proof. Let g and f be two different elements in Φ. Then there exists no class containing two sequences $\{g_n\}$ and $\{f_n\}$ with $g_n=g$, $f_n=f$ for every n.

Because if it is not true, $\{g_n\}$ and $\{f_n\}$ are equivalent. And then there exists a fundamental sequence of neighbourhoods $\{V_{r_i}(0,r_i,m_i)\}$ such that $g_i-f_i\in V_{r_i}(0,r_i,m_i)$ for every i, that is, $g-f\in V_{r_i}(0,r_i,m_i)$ for every i. This implies $g=f$ by Lemma 8 in [4].

Next, we shall prove that $g\in V_i\,(0,r,m)$ implies $\hat{g}\in\hat{V}_i\,(0,r,m)$. Since we have $V_i\,(0,1,m)=U_i\,(0,\varepsilon_i,m)$ by the paper [4], we obtain $V_i\,(0,r,m)=U_i\,(0,r\varepsilon_i,m)$. Hence we have

$$\left\|\sum_{k=1}^{m}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})\varphi_{k,n_{i-1}}\right\|<r\varepsilon_i.$$

Then there exists some number r', $0<r'<r$ such that

$$\left\|\sum_{k=1}^{m}\lambda_{k,n_{i-1},n_i}(g,\varphi_{k,n_i})\varphi_{k,n_{i-1}}\right\|<r'\varepsilon_i<r\varepsilon_i.$$

Consequently we obtain $g\in V_i\,(0,r',m)$. By Definition 5, this shows

$$\hat{g}\in\hat{V}_i\,(0,r,m).$$

Conversely, if we have $\hat{g}\in\hat{V}_i\,(0,r,m)$, there exist some number r', $0<r'<r$ and some integer N such that

$$g_n=g\in V_i\,(0,r',m)\qquad\text{if } n\geqq N.$$

And then we obtain $g\in V_i\,(0,r,m)$.

Theorem 2. *The set $\hat{\Phi}_0$ is dense in $\hat{\Phi}$.*

Proof. Let $\hat{g}$ be any element in $\hat{\Phi}$. And let an R-Cauchy sequence $\{g_n\}$ belong to $\hat{g}$. Then there exists a fundamental sequence of neighbourhoods $\{V_{r_i}(0,r_i,m_i)\}$, such that the relations $n\geqq i$ and $m\geqq i$ imply

$$g_n-g_m\in V_{r_i}\,(0,r_i,m_i).$$

Let $\hat{g}_n$ be the class containing the reptitive sequence $g_n, g_n, \cdots$.