189. On the Bessel Kernel for a Domain

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1. Aronszajn and Smith [2] developed the theory of Bessel potentials from the standpoints of their functional spaces and functional completions [1]. Let $P^{\alpha}(R^n)$ be the functional completion of $C_0^{\infty}(R^n)$ with the norm $\|u\|_{\alpha}^2=\int(1+|\xi|^2)^{\alpha}|\hat{u}(\xi)|^2\,d\xi$. They showed that $P^{\alpha}(R^n)$ has the reproducing kernel $G_{2\alpha}(x-y)$ determined by

$$G_{2\alpha}(x)=\frac{1}{2^{(n+2\alpha-2)/2}\pi^{n/2}\Gamma(\alpha)}K_{(n-2\alpha)/2}(|x|)|x|^{(2\alpha-n)/2}$$

where $K_{(n-2\alpha)/2}$ is the modified Bessel function of third kind. The purpose of this paper is to consider the kernel of functional completion $P^{\alpha}(\Omega)$ of $C_0^{\infty}(\Omega)$ with the norm $\|u\|_{\alpha}$. Making use of the methods of general balayage and the theory of α-harmonic functions introduced by M. Itô [6], we define the Green function $G_{2\alpha}^{\Omega}(x,y)$ and α-harmonic functions in the theory of Bessel potentials. Let $E_{2\alpha}(\Omega)$ be the class of all positive measures of finite energy with compact support contained in Ω, $U_{2\alpha}^{\mu}$ be the potential of $\mu\in E_{2\alpha}(\Omega)$ in the functional space $P^{\alpha}(\Omega)$ and $G_{2\alpha}^{\Omega}\mu$ (resp. $\check{G}_{2\alpha}^{\Omega}\mu$) be the potential of μ with respect to the kernel $G_{2\alpha}^{\Omega}(x,y)$(resp. $\check{G}_{2\alpha}^{\Omega}(x,y)=G_{2\alpha}^{\Omega}(y,x)$). We shall prove the following results.

(1) Let Ω be a domain in R^n. Then for every $\mu\in E_{2\alpha}(\Omega)$, there exists an α-harmonic function $H_{2\alpha}^{\mu}(x)$ in Ω such that

$$U_{2\alpha}^{\mu}(x)=G_{2\alpha}^{\Omega}\mu(x)+H_{2\alpha}^{\mu}(x).$$

(2) The following conditions are equivalent:

(a) There exists a bounded domain $\Omega(\neq\emptyset)$ in R^n such that the Green function $G_{2\alpha}^{\Omega}(x,y)$ is the kernel of the functional space $P^{\alpha}(\Omega)$ i.e., $U_{2\alpha}^{\mu}=G_{2\alpha}^{\Omega}\mu$ in $P^{\alpha}(\Omega)$ for every $\mu\in E_{2\alpha}(\Omega)$.

(b) There exist a bounded domain Ω in R^n and a measure $\mu(\neq 0)\in E_{2\alpha}(\Omega)$ such that $G_{2\alpha}^{\Omega}\mu\in P^{\alpha}(\Omega)$ and $G_{2\alpha}^{\Omega}\mu=\check{G}_{2\alpha}^{\Omega}\mu$ in $P^{\alpha}(\Omega)$.

(c) $0<\alpha\leqq 1$.

2. According to Aronszajn and Smith [2], we define the Bessel potentials and summarize the results obtained in [2].

Definition 1. The Bessel potential of order 2α, $\alpha>0$, of a positive measure μ is defined by $G_{2\alpha}\mu(x)=\int G_{2\alpha}(x-y)d\mu(y)$. We denote by $E_{2\alpha}(R^n)$ the class of all positive measures for which the 2α-energy