183. On Weakly Compact Spaces

By Masao SAKAI

(Comm. by Kinjirô KUNUGI, M. J. A., April 12, 1971)

A topological space S is said to be *AU-weakly compact*, if every countable open covering of S contains a finite subfamily whose union is deuse in S, and S is said to be *MP-weakly compact*, if every pairwise disjoint infinite family of open sets O_α, $\alpha \in A$, has a point $p \in S$ whose every neighbourhood meets infinitely many O_α. The point p is called a *cluster point* of the family $\{O_\alpha\}_{\alpha \in A}$. K. Iseki [1] [2] [3] and S. Kasahara [4] proved the following:

Proposition. *The following properties of a regular space S are equivalent:*

(1) *S is AU-weakly compact.*

(2) *S is MP-weakly compact.*

(3) *Every locally finite family of open sets O_α contains a finite subfamily whose union covers the union of all O_α.*

(4) *Every locally finite open covering of S contains a finite subcovering.*

We shall prove only that (2)→(3) using the following:

Lemma. *Every point-finite covering of a topological space contains an irreducible subcovering.*

This lemma was proved by R. Arens and J. Dugundji [5].

Proof that (2)→(3). Let S be a regular MP-weakly compact space and let $\{O_\alpha\}_{\alpha \in A}$ be a locally finite family of open sets of S. By the lemma, there is an irreducible subfamily $\{O_\beta\}_{\beta \in B}$ such that $\bigcup_{\beta \in B} O_\beta = \bigcup_{\alpha \in A} O_\alpha$. We shall prove that B is a finite set. Let us assume that B is an infinite set. By the irreducibility of $\{O_\beta\}_{\beta \in B}$ for every $\beta \in B$, $O_\beta - \bigcup_{\gamma \in B-\{\beta\}} O_\gamma$ is non-empty, then it contains a point p_β such that $p_\beta \in O_\beta$ and $p_\beta \bar{\in} O_\gamma$, $\gamma \in B-\{\beta\}$. By the regularity of the space S, every p_β has an open neighbourhood V_β such that $\bar{V}_\beta \subset O_\beta$. It is easily seen that for every $\beta \in B$ $p_\beta \in V_\beta$ and $p_\beta \bar{\in} \bar{V}_\gamma$, $\gamma \in B-\{\beta\}$. By the locally finiteness of $\{O_\beta\}_{\beta \in B}$, $\bigcup_{\gamma \in B-\{\beta\}} \bar{V}_\gamma$ is closed, then $W_\beta = V_\beta - \bigcup_{\gamma \in B-\{\beta\}} \bar{V}_\gamma$ is open and contains p_β. It is obvious that the open infinite family $\{W_\beta\}_{\beta \in B}$ is pairwise disjoint and locally finite. By the property (2), the family $\{W_\beta\}_{\beta \in B}$ has at least one cluster point, contrary to the locally finiteness of the family $\{W_\beta\}_{\beta \in B}$. Then B must be a finite set and the proof of (2)→(3) is completed.

Let S be a topological space. Each family of regularly closed sets $\bar{O}_\alpha$, $\alpha \in A$, of S is called a *regularly closed family*, and each covering of S