221. *Note on the Asymptotic Normality of a Stochastic Process with Independent Increments*

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1. In the present paper we are concerned with the continuous analogue of the classical central limit theorem. The Lindeberg theorem establishes necessary and sufficient conditions under which sums of mutually independent random variables are asymptotically normally distributed. We shall show that the normal convergence law of the Lindeberg type holds for a stochastic process with independent increments, which is essentially the continuous parameter version of a sequence of consecutive sums of mutually independent random variables. In some practical applications, it is of real importance to determine limiting distributions for continuous parameter processes with independent increments [1] [2].

2. Let $\{x_t, t\geq 0\}$ be a continuous parameter process with independent increments which is not necessarily temporally homogeneous. In what follows, we assume that $x_0\equiv 0$ and that there are no fixed points of discontinuity. As is well known [3], the characteristic function of x_t has the form

$$(1)\qquad \begin{cases} E(e^{i\zeta x_t})=e^{\varphi(\zeta,t)}, \\ \varphi(\zeta, t)=i\zeta m(t)-\dfrac{\zeta^2}{2}v^2(t)+\displaystyle\int_{-\infty}^{\infty}\left(e^{i\zeta u}-1-\frac{i\zeta u}{1+u^2}\right)\nu_t(du). \end{cases}$$

Here $m(t)$ is a continuous function of t, $v^2(t)$ is a non-negative, monotone non-decreasing, and continuous function of t, and $\nu_t(du) \equiv\int_{\tau=0}^{t}\nu(d\tau du)$ is a measure on $(-\infty, \infty)\backslash\{0\}$ satisfying $\nu(\{t\}\times du)\equiv 0$ and

$$(2)\qquad \int_{-\infty}^{\infty}\frac{u^2}{1+u^2}\nu_t(du)<\infty.$$

The following lemma can be verified directly from the formula (1).

Lemma 1. *If $E(x_t^2)$ is finite, or equivalently,*

$$(3)\qquad \int_{-\infty}^{\infty}u^2\nu_t(du)<\infty$$

for all $t>0$, then the expectation $\mu(t)=E(x_t)$ and the variance $\sigma^2(t)=\mathrm{Var}\,(x_t)$ are given by

$$(4)\qquad \mu(t)=m(t)+\int_{-\infty}^{\infty}\frac{u^3}{1+u^2}\nu_t(du),\qquad \sigma^2(t)=v^2(t)+\int_{-\infty}^{\infty}u^2\nu_t(du).$$