218. *On a Theorem of Bouldin*

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1. Introduction. In this paper A means a bounded linear operator on a complex Hilbert space $\mathfrak{H}$ and the numerical range $W(A)$ means the set of complex numbers

$$W(A)=\{(Ax\,|\,x)\,;\,x\in\mathfrak{H},\ \|x\|=1\}.$$

It is well known that $W(A)$ is convex and the closure $\overline{W(A)}$ of $W(A)$ contains the spectrum $\sigma(A)$ of A.

Recently, R. Bouldin [2] [3] has shown an elegant result which determines the numerical range of the product AB in terms of $\overline{W(A)}$ and $\overline{W(B)}$ under a suitable condition (cf. Theorem 2 in the below).

The purpose of the present paper is to give an elementary proof of Bouldin's theorem.

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2. Bouldin's theorem. We shall show at first the following theorem by a simple calculation:

Theorem 1. *If A is an operator and B is nonnegative then $W(B^{1/2}AB^{1/2})\subset W(A)W(B)$.*

Proof. If $B^{1/2}x\neq 0$ for a unit vector, then

$$\begin{aligned}(B^{1/2}AB^{1/2}x\,|\,x)&=(AB^{1/2}x\,|\,B^{1/2}x)\\&=\|B^{1/2}x\|^2\left(A\frac{B^{1/2}x}{\|B^{1/2}x\|}\,\middle|\,\frac{B^{1/2}x}{\|B^{1/2}x\|}\right)\\&=(Bx\,|\,x)\left(A\frac{B^{1/2}x}{\|B^{1/2}x\|}\,\middle|\,\frac{B^{1/2}x}{\|B^{1/2}x\|}\right).\end{aligned}$$

If $B^{1/2}x=0$, then

$$(B^{1/2}AB^{1/2}x\,|\,x)=0=(Bx\,|\,x).$$

Therefore the theorem is proved.

Remark. Since

$$\sigma(AB)=\sigma(B^{1/2}AB^{1/2})\pm\{0\},$$

Theorem 1 implies at once

$$\sigma(AB)\subset\overline{W(A)}\,\overline{W(B)}$$

under the same conditions of Theorem 1.

The above considerations imply at once the following theorem which is originally obtained by a help of perturbation theory:

Theorem 2 (Bouldin). *If A is an operator and B is nonnegative then*