214. Localization Principle for Differential Complexes and Its Application

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0. In this note we announce a general principle for proving the exactness or the partial exactness, in sheaf level, of complexes with first order differential operators as their differentiations, and its application to the Dolbeault type sequences for real submanifolds in complex manifolds.

Throughout this note we assume the differentiability of class C^∞ for manifolds, vector bundles, differential operators and so on, unless otherwise stated. When F is a vector bundle over a manifold M, $C^\infty(F)$ denotes the set of smooth sections of F over M, while $C^\infty(U,F)$ denotes the set of smooth sections of F over an open subset U of M.

1. Localization principle. We shall first fix a differential complex

$$\cdots \longrightarrow E^{i-1} \xrightarrow{\partial^{i-1}} E^i \xrightarrow{\partial^i} E^{i+1} \longrightarrow \cdots$$

where E^i, $i \in \boldsymbol{Z}$ are (complex) vector bundles over a manifold M, and $\partial^i : E^i \to E^{i+1}$ are first order differential operators such that $\partial^{i+1} \cdot \partial^i = 0$. For brevity we denote this complex by $E^\cdot$ and the corresponding complex for sections $(C^\infty(E^i))_{i\in \boldsymbol{Z}}$ by $C^\infty(E^\cdot)$. Now we shall list up the definitions which are needed to formulate the fundamental theorems.

Definition 1. A subcomplex $A^\cdot = (A^i)_{i\in\boldsymbol{Z}}$ of $C^\infty(E^\cdot)$ is called *complete* if and only if (1) A^i, $i \in \boldsymbol{Z}$ are complete locally convex topological vector spaces such that the inclusions $A^i \hookrightarrow C^\infty(E^i)$ are continuous and (2) $\partial^i | A^i : A^i \to A^{i+1}$ is continuous for these topologies.

Definition 2. Let $A^\cdot$, $B^\cdot$ be two subcomplexes of $C^\infty(E^\cdot)$ such that $A^\cdot \subset B^\cdot$ and let s^i, $i \leqq q$ be linear maps from A^i to B^{i-1}. The family of maps $s = (s^i)_{i\leqq q}$ is said to be a $(-\infty, q)$-*homotopy for the inclusion* $A^\cdot \hookrightarrow B^\cdot$ if and only if (1) $(\partial^{i-1}s^i + s^{i+1}\partial^i)u = u$ for $i < q$ and for $u \in A^i$, and (2) $\partial^{q-1}s^q u = u$ for $u \in A^q$ such that $\partial^q u = 0$. In case $A^\cdot$, $B^\cdot$ are both complete, s is said to be continuous when each map s^i, $i \leqq q$ is continuous.

Definition 3. Let $A^\cdot \subset B^\cdot \subset C.$ be subcomplexes of $C^\infty(E^\cdot)$ such that $A^\cdot \subset B^\cdot \subset C^\cdot$. Let further s^i, $i > q$ be linear maps from C^i into C^{i-1} and s^q a linear maps from C^q into $C^\infty(E^{q-1})$. The family of maps $s = (s^i)_{i\geqq q}$ is called a (q, ∞)-*homotopy of* $C^\cdot$ *modulo* $(A^\cdot, B^\cdot)$ if and only if (1) $s^i(A^i) \subset B^{i-1}$ for $i > q$ and (2) $(\partial^{i-1}s^i + s^{i+1}\partial^i)u \equiv u$ mod. $B^\cdot$ for $i \geqq q$ and for $u \in C^i$. In case $C^\cdot$ is a complete subcomplex and each s^i, $i \geqq q$