208. *A Remark on Nowhere Differentiability of Sample Functions of Gaussian Processes*

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1. The Result. Let $\{X(t,\omega)\,;\,0\leq t\leq 1\}$ be a Gaussian process with the mean $E[X(t)]=0$. We assume in this paper that the process has continuous sample functions and there exists an even, non-negative, and non-decreasing function $\varphi(h)$ such that

$$E[(X(t+h)-X(t))^2]\geq \varphi^2(h).$$

The sample functions of the Brownian motion are, as well known, nowhere differentiable with probability one [5]. J. Yeh [6] proved that the sample functions are almost nowhere differentiable with probability one, for the Gaussian processes satisfying the condition $\lim_{h\downarrow 0} h/\varphi(h)=0$. But there exists a gap between the property of almost nowhere differentiability with probability one and the property of nowhere differentiability with probability one, because almost nowhere differentiability is essentially non differentiability at each point with probability one, but this does not yield nowhere differentiability with probability one.

In this paper we shall prove a theorem about nowhere differentiability of sample functions as follows:

Theorem. *If there exists a positive integer q such that*

$$\lim_{h\downarrow 0}\left(\frac{h}{\varphi(h)}\right)^q\frac{1}{h}=0,$$

and if there exists a positive integer p such that

$$\overline{\lim_{h\downarrow 0}}\sup_{|t-s|\geq ph}|E[\Delta_h Y(t)\Delta_h Y(s)]|\leq 1/(2q),$$

where

$$\Delta_h Y(t)=(X(t+h)-X(t))/\{E[(X(t+h)-X(t))^2]\}^{1/2},$$

then we have

$$P\left[\overline{\lim_{h\downarrow 0}}\frac{|X(t+h)-X(t)|}{h}=+\infty \qquad \text{for all } t,\ 0\leq t<1\right]=1.$$

Recently, S. M. Berman has proved nicely the same fact for some class of Gaussian processes by means of local times of the stochastic processes [1][2]. Our class which satisfies the conditions of Theorem is not contained in Berman's class, and the proof of our Theorem is much simpler, making use of the idea of [3].

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