205. *On a Non-linear Volterra Integral Equation with Singular Kernel*

By Takashi KANAZAWA

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In the present paper we consider the solution $y(x)$ of the non-linear Volterra integral equation

$$y(x)=f(x)+\int_0^x p(x,t)k(x,t,y(t))dt \tag{1}$$

where $p(x,t)$ is supposed to be unbounded in the region of integration.

Examples. $p(x,t)=(x-t)^{-1/2}$, or $p(x,t)=t(x^2-t^2)^{-1/2}$.

Evans [1] studied a similar problem using the convolution. Our treatment below is more elementary than his. We also consider the continuity and differentiability with respect to a parameter of solutions of (1) when it contains a parameter.

1. Existence theorem. In equation (1) we shall assume the four conditions:

(a) $f(x)$ is continuous in the interval I_a,
$I_a=\{x\,|\,0\leqslant x\leqslant a\}$;

(b) $k(x,t,y)$ is continuous in the region Δ,
where $\Delta=\{(x,t,y)\,|\,0\leqslant t\leqslant x\leqslant a,\ |y-f(x)|\leqslant b\}$,
$\sup_{0\leqslant t\leqslant x\leqslant a} k(x,t,f(x))=K$,
$k(x,t,y)$ satisfies a Lipschitz condition:
$|k(x,t,y_1)-k(x,t,y_2)|\leqslant L|y_1-y_2|$;

(c) $\int_0^x |p(x,t)|\,dt\leqslant M<\infty \qquad (0\leqslant x\leqslant a)$;

(d) for any $\varepsilon>0$, there exists $\delta>0$, independent of x and α, such that

$$\int_\alpha^{\alpha+\delta} |p(x,t)|\,dt<\varepsilon \qquad \text{for all } 0\leqslant\alpha\leqslant x-\delta.$$

Theorem 1. *Under the conditions* (a), (b), (c), (d), *equation* (1) *has a unique continuous solution on the interval* $0\leqslant x\leqslant h$, *where* h *is determind as follows*:

for any ρ, $0<\rho<1$, *let* $P=\min\left(\frac{\rho}{L},\frac{b}{K}\right)$ *and then let* $h=\min(r,a)$,

where r *is determined by*

$$\int_0^x |p(x,t)|\,dt\leqslant P \qquad (0\leqslant x\leqslant r).$$

Proof. For $n=1,2,\cdots$, let us put