34. Note on Dirichlet Series. XII. On the Analogy between Singularities and Order-Directions. I

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(Comm. by Z. SUETUNA, M.J.A., March 12, 1954)

(1) **Introduction.** Let us put

$$(1.1)\quad F(s)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n s)\quad (s=\sigma+it,\ 0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n\to+\infty).$$

C. Biggeri has proved the next theorem.

C. Biggeri's Theorem (1) pp. 979–980, 2) p. 294). *Let (1.1) be simply convergent for $\sigma>0$. If $\mathfrak{R}(a_n)\geqq 0$ $(n=1, 2,\dots)$ and $\lim_{n\to+\infty}(\cos(\arg(a_n)))^{1/\lambda_n}=1$, then $s=0$ is the singular point.*

In this note, we shall establish an analogous theorem concerning order-direction. We begin with

Definition. *Let (1.1) be uniformly convergent in the whole plane. Then, we call the direction $\mathfrak{J}(s)=t$ the order-direction of (1.1), provided that, in $|\mathfrak{J}(s)-t|\leqq\varepsilon$ (ε: any positive constant), (1.1) has the same order as in the whole plane, i.e.*

$$\overline{\lim_{\sigma\to-\infty}} 1/(-\sigma).\ \log^+\log^+ M(\sigma)=\overline{\lim_{\sigma\to-\infty}} 1/(-\sigma).\ \log^+\log^+ M(\sigma, t, \varepsilon),$$

where $M(\sigma)=\operatorname{Sup}_{-\infty<t<+\infty}|F(\sigma+it)|$, $\quad M(\sigma, t, \varepsilon)=\operatorname{Max}_{\mathfrak{R}(s)=\sigma,\ |\mathfrak{J}(s)-t|\leqq\varepsilon}|F(s)|$,

$\log^+ x = \operatorname{Max}\{0, \log x\}$.

Remark. The order-direction is a special case of the order-curve defined in the previous note (3)).

Our theorem is the following

Theorem. *Let (1.1) be uniformly convergent in the whole plane. If we have*

$$(1.2)\quad \begin{aligned} &(\mathrm{i})\quad \mathfrak{R}(a_n)\geqq 0\quad (n=1, 2,\dots),\\ &(\mathrm{ii})\quad \lim_{n\to\infty} 1/\lambda_n \log\lambda_n\cdot\log(\cos\theta_n)=0,\quad \arg(a_n)=\theta_n, \end{aligned}$$

then $\mathfrak{J}(s)=0$ is the order-direction of (1.1).

As its corollary, we get

Corollary. *Let (1.1) with $\mathfrak{R}(a_n)\geqq 0$ $(n=1, 2,\dots)$, $\lim_{n\to\infty}(\cos\theta_n)^{1/\lambda_n}=1$, $(\theta_n=\arg(a_n))$ be simply (necessarily absolutely) convergent in the whole plane. Then $\mathfrak{J}(s)=0$ is the order-direction of (1.1). In particular, if $|\theta_n|\leqq\theta<\pi/2$ $(n=1,2,\dots)$, the same conclusion holds.*

(2) **Lemmas.** To prove this theorem, we need some lemmas.

Lemma I (C. Tanaka, 4) p. 77, corollary IV). *Under the same assumptions as in our Theorem, the order ρ of (1.1) is given by*