75. Ergodic Decomposition of Stationary Linear Functional*)

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(Comm. by K. KUNUGI, M.J.A., May 13, 1954)

In this note, we shall prove ergodic decomposition of stationary semi-trace of a separable D^*-algebra with a motion, applying the reduction theory of von Neumann [2][1] and a decomposition of a two-sided representation [3]. The theorem in this paper contains the ergodic decompositions of stationary trace on separable C^*-algebra with a motion and the ergodic decomposition of invariant regular measure on separable locally compact Hausdorff space with a group of homeomorphisms. (Cf. Th. 4 and Th. 7 of [3].)

Let $\mathfrak{A}$ be a D^*-algebra (: normed $*$-algebra over the complex number field) with an approximate identity $\{e_\alpha\}$ and with a motion G where G is meant by any group of isometric *automorphisms on $\mathfrak{A}$. (Cf. [3].) Let τ be a G-stationary semi-trace of $\mathfrak{A}$, i.e. τ is a linear functional on $\mathfrak{A}^2$ (=self-adjoint (s.a.) subalgebra generated by $\{xy\,;\, x, y\in\mathfrak{A}\}$) such that $\tau(x^*x)\geqq 0$, $\tau(xy)=\tau(yx)=\overline{\tau(x^*y^*)}$, $\tau((xy)^*xy)\leqq\|x\|^2\tau(y^*y)$, $\tau((e_\alpha x)^*e_\alpha x)\xrightarrow[\alpha]{}\tau(x^*x)$ and $\tau(x^sy^s)=\tau(xy)$ for all $x, y\in\mathfrak{A}$ and $s\in G$. Putting $\mathfrak{N}=\{x\,;\,\tau(x^*x)=0,\ x\in\mathfrak{A}\}$, $\mathfrak{N}$ is a two-sided ideal in $\mathfrak{A}$. Let $\mathfrak{A}^\theta$ be the quotient algebra $\mathfrak{A}/\mathfrak{N}$ and x^θ the class ($\in\mathfrak{A}^\theta$) containing x which is an incomplete Hilbert space with inner product $(x^\theta, y^\theta)=\tau(y^*x)$. Let $\mathfrak{H}$ be the completion of $\mathfrak{A}^\theta$ with respect to the norm $\|y^\theta\|\,(=\tau(y^*y)^{1/2})$. Putting $x^ay^\theta=(xy)^\theta$, $x^by^\theta=(yx)^\theta$, $jy^\theta=y^{*\theta}$ and $U_sy^\theta=y^{s\theta}$ for all $x, y\in\mathfrak{A}$ and $s\in G$, $\{x^a, x^b, j, \mathfrak{H}\}$ defines a two-sided representation of $\mathfrak{A}$. (Cf. [3].) Moreover $\{U_s, \mathfrak{H}\}$ defines a dual unitary representation of G. Indeed, for any $x, y\in\mathfrak{A}$ $(U_sy^\theta, U_sy^\theta)=(x^{s\theta}, y^{s\theta})=\tau(y^sx^{*s})=(y^\theta, x^\theta)$ and $U_{st}y^\theta=y^{st\theta}=U_ty^{s\theta}=U_tU_sy^\theta$. Hence U_s has uniquely unitary extension on $\mathfrak{H}$ which satisfies the required relations. These representations are uniquely determined by the given τ within unitary equivalence. (Cf. [3].)

For any collection F of bounded operators and two W^*-algebras W_1, W_2 on a Hilbert space, we denote F' the collection of all bounded operators commuting for all $A\in F$ and $W_1\smile W_2$ the W^*-algebra generated by W_1 and W_2.

Let W^a, W^b and W_G be W^*-algebras generated by $\{x^a;\ x\in\mathfrak{A}\}$, $\{x^b;\ x\in\mathfrak{A}\}$ and $\{U_s;\ s\in G\}$ respectively, then $W^a=W^{b\prime}$ and $jAj=A^*$ for all $A\in W^a\frown W^b$. (Cf. Th. 2 of [3].)

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*) This paper is a continuation of the previous paper [3].

1) Numbers in brackets refer to the references at the end of this paper.