93. *Note on Linear Topological Spaces*

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§1. The purpose of this note is to give a generalization of Y. Kawada's theorem to convex linear topological spaces and some related remarks. As we treat only separative convex linear topological spaces, we shall call them merely *convex spaces*.

In the sequel, the word "Isomorphism" means always algebraic isomorphism together with homeomorphism, unless the contrary is mentioned. For two convex spaces E and F, $L(E, F)$ is the space of all continuous linear mappings of E to F. For any subset A of E and B of F, $(A \mid B)$ denotes the set; $\{u;\ u \in L(E, F)\ u(A) \subseteq B\}$. For a family $\mathfrak{S}$ of bounded subsets of E such that for any $A_1 \in \mathfrak{S}$ and $A_2 \in \mathfrak{S}$, there exists an $A_3 \in \mathfrak{S}$ with $A_1 \smile A_2 \subseteq A_3$ and $\bigcap_{A \in \mathfrak{S}} A = E$, we can define a convex linear topology in $L(E, F)$ whose basis of neighborhood of the origin consists of all $(A \mid V)$ where $A \in \mathfrak{S}$ and V is a neighborhood of o in F. This topology is called $\mathfrak{S}$-*topology*. We write $\langle x, x' \rangle$ instead of $x'(x)$ $(x \in E,\ x' \in E')$ where E' denotes the conjugate space of E. A neighborhood of the origin is called an *o-neighborhood*.

§2. Theorem 1. (Kawada[1]) *Let E and F be two convex spaces. If $L(E, E)$ and $L(F, F)$ are algebraically (ring) isomorphic, then there exists an algebraic isomorphic mapping φ of E onto F and $\tilde{\varphi}$ of E' onto F' such that* $\langle x, x' \rangle = \langle \varphi(x), \tilde{\varphi}(x') \rangle$ $(x \in E,\ x' \in E')$.

Proof. We sketch Kawada's proof.

(a) Any minimal left ideal $\mathfrak{A}$ of $L(E, E)$ is algebraically isomorphic to E in the manner $E \ni x \longleftrightarrow u_x \in \mathfrak{A}$ implies $v(x) \longleftrightarrow v \cdot u_x$ $(v \in L(E, E))$.

In fact, there exists an element x_0 of E and $u_0 \in \mathfrak{A}$ with $u_0(x_0) \neq 0$. We can easily see that the linear mapping $u \to u(x_0)$ maps $\mathfrak{A}$ onto E. The set $\{u;\ u \in \mathfrak{A}\ u(x_0) = 0\}$ is a left ideal contained in $\mathfrak{A}$ and not identical to it, so a zero ideal, because of the minimalness of $\mathfrak{A}$. Thus this mapping is an expected algebraic isomorphism. Conversely, the set $\{u_y;\ u_y(x) = \langle x, x_0' \rangle y,\ y \in E\}$ for non-zero $x_0' \in E'$ is a minimal left ideal.

(b) Let Φ be the given algebraic isomorphic mapping of $L(E, E)$ onto $L(F, F)$. Then

$$E \ni x \longleftrightarrow u_x \in \mathfrak{A} \longleftrightarrow \tilde{u}_{\tilde{x}} \in \tilde{\mathfrak{A}} \longleftrightarrow \tilde{x} \in F$$

and

$$v(x) \longleftrightarrow v \cdot u_x \longleftrightarrow \Phi(v)\tilde{u}_{\tilde{x}} \longleftrightarrow \Phi(v)[\tilde{x}]$$

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1) Y. Kawada: "Ueber den Operatorenring Banachscher Räume", Proc. Imp. Acad., **19**, 616-621 (1943).