148. *Uniform Convergence of Fourier Series. II*

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1. A. Zygmund has proved the following.

Theorem 1. *Let* $0<\alpha<1$. *If* $f(x)$ *is continuous and*

$$\omega(1/n)=o(1/n^{\alpha}),$$

then the Fourier series of $f(x)$ *is summable* $(C,-\alpha)$ *uniformly.*

This theorem was generalized by S. Izumi and T. Kawata [1] and S. Izumi [2]. We give another generalization of Theorem 1. In our theorem, the case where the modulus of continuity is of order $o(1/(\log n)^{\beta})$ is contained. (See Cor. 2.) The method of proof is analogous to [3]. (Cf. [4].)

2. **Theorem 2.** *If* $f(x)$ *is of class* $\phi(n)$,[1] $\phi(n)$ *being less than* n, *and is continuous with the modulus of continuity* $\omega(\delta)$, *then*[2]

$$|\sigma_n^{-\alpha}(x)-f(x)|\leqq C\left[\omega\left(\frac{1}{n}\right)^{1-\alpha}\left(\frac{n}{\phi(n)}\right)^{\alpha}+\frac{1}{n}\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}dt\right],$$

where $0<\alpha<1$ *and* $\sigma_n^{-\alpha}(x)$ *is the nth Cesàro mean of the Fourier series of* $f(x)$ *of order* $-\alpha$.

Proof. We have

$$\sigma_n^{-\alpha}(x)-f(x)=\int_0^{\pi}\varphi_x(t)K_n^{-\alpha}(t)dt=\left[\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right]\varphi_x(t)K_n^{-\alpha}(t)dt=I+J$$

say, where $K_n^{-\alpha}(t)$ is the Fejér kernel of order $-\alpha$, and $\varphi_x(t)=f(x+t)+f(x-t)-2f(x)$. It is known that

$$(1)\qquad K_n^{-\alpha}(t)=\psi_n^{-\alpha}(t)+r_n^{-\alpha}(t)$$

where

$$(2)\qquad \psi_n^{-\alpha}(t)=\cos\left(\left(n+\frac{1-\alpha}{2}\right)t-\frac{1-\alpha}{2}\pi\right)\Big/A_n^{-\alpha}\left(2\sin\frac{t}{2}\right)^{1-\alpha},$$

$$(3)\qquad r_n^{-\alpha}(t)=O(1/nt^2),\qquad |K_n^{-\alpha}(t)|\leqq Cn.$$

Then we get by (3)

$$I\leqq\int_0^{\pi/n}|\varphi_x(t)|\,|K_n^{-\alpha}(t)|\,dt\leqq Cn\int_0^{\pi/n}|\varphi_x(t)|\,dt\leqq Cn\omega\left(\frac{\pi}{n}\right)\int_0^{\pi/n}dt=C\omega\left(\frac{1}{n}\right).$$

1) A function $f(x)$ is said to be of class $\phi(n)$ if $\phi(n)\uparrow\infty$ as $n\to\infty$ and

$$\int_a^b f(x+t)\cos nt\,dt=O(1/\phi(n))$$

uniformly for all x, n, a, b with $b-a\leqq 2\pi$. (Cf. [4].) If $\omega(1/n)\leqq 1/\phi(n)$, then the condition becomes trivial, and hence we may suppose that $\omega(1/n)\geqq 1/\phi(n)$.

2) C denotes an absolute constant, which need not be equal in each occurrence.